

Mathematics mind map

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Number and algebra

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Calculus

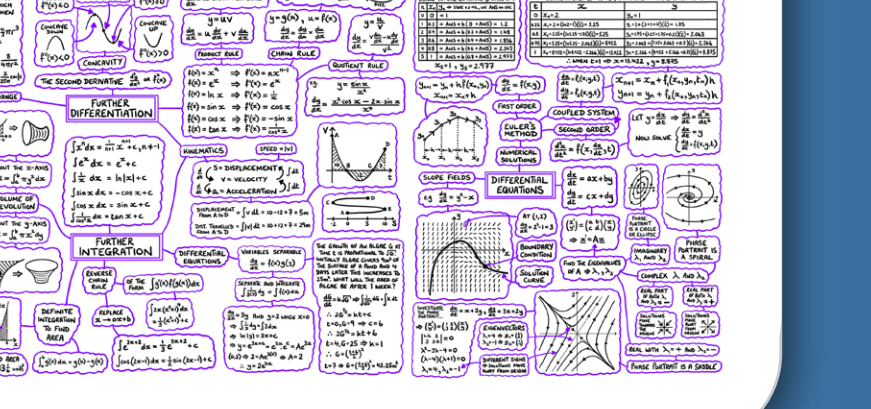
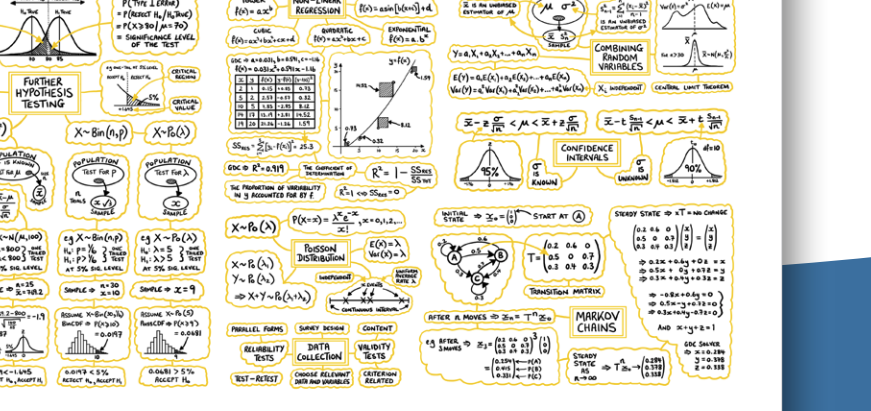
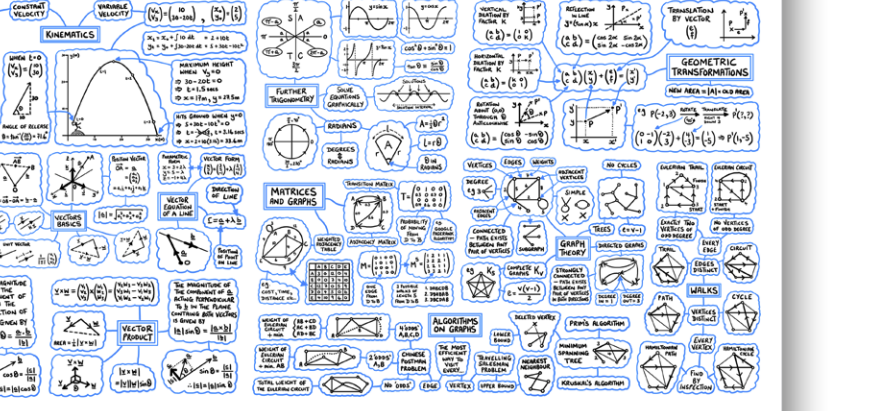
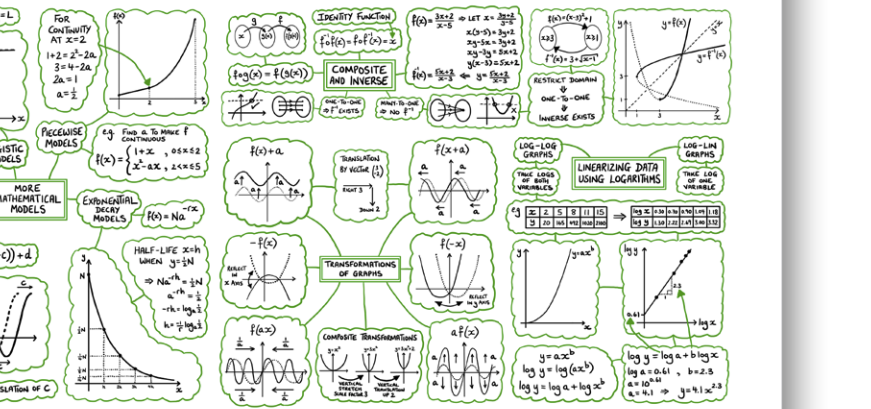
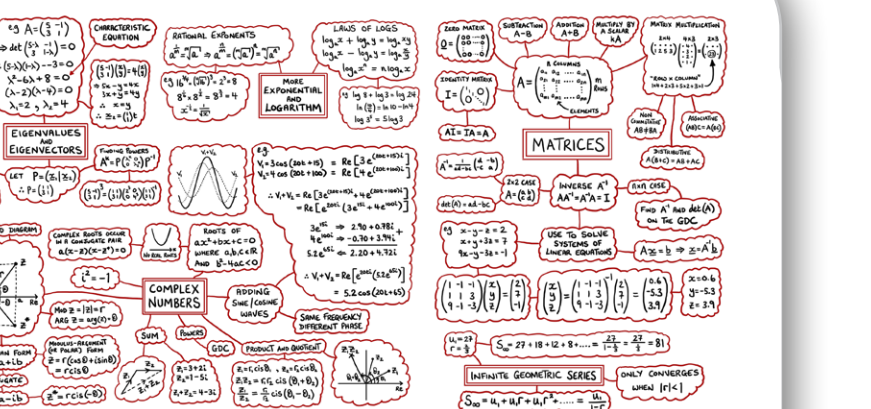
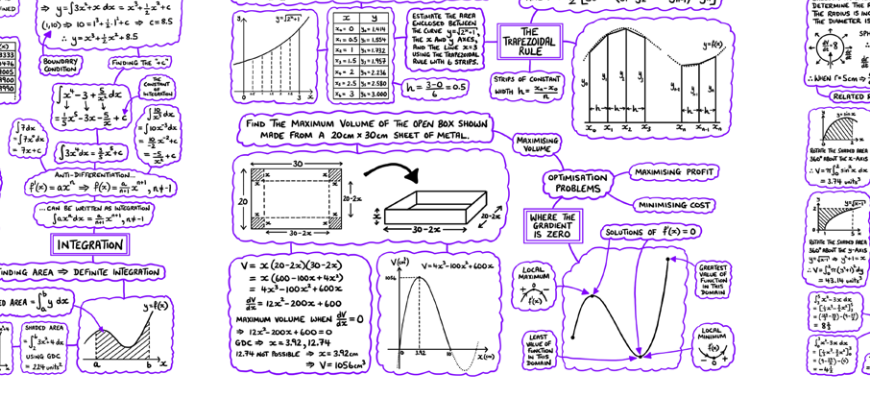
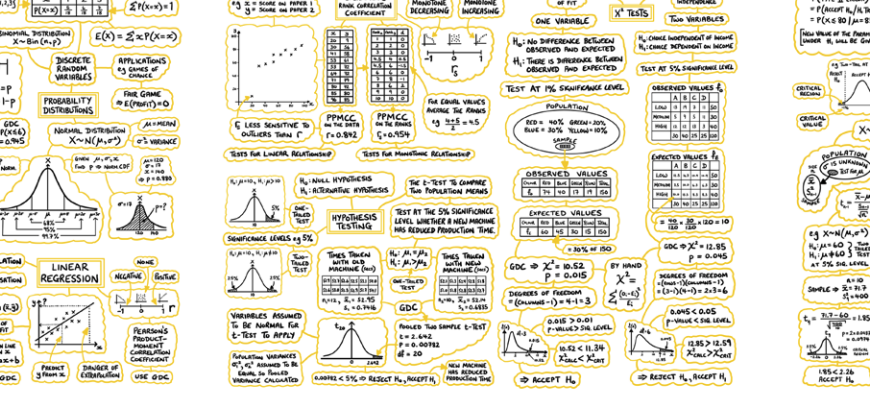
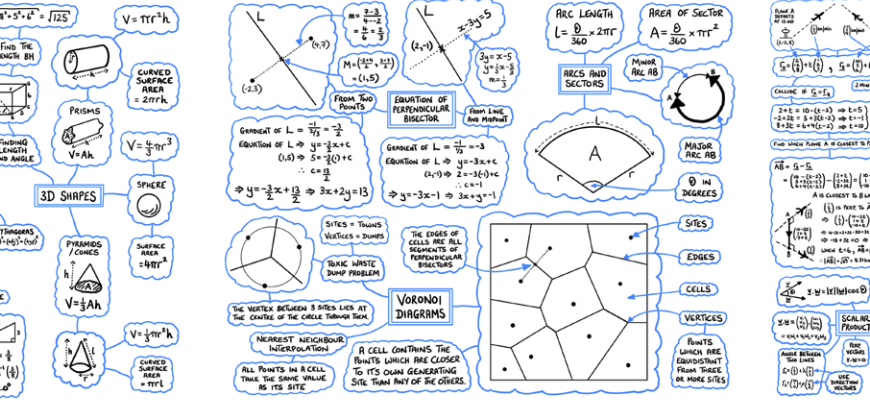
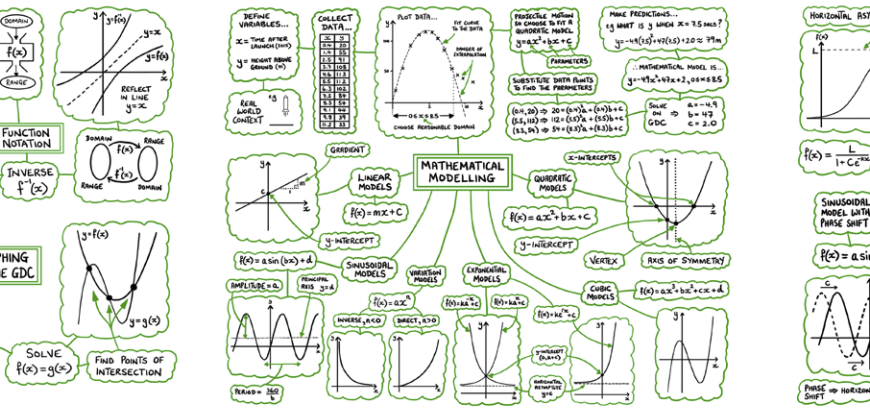
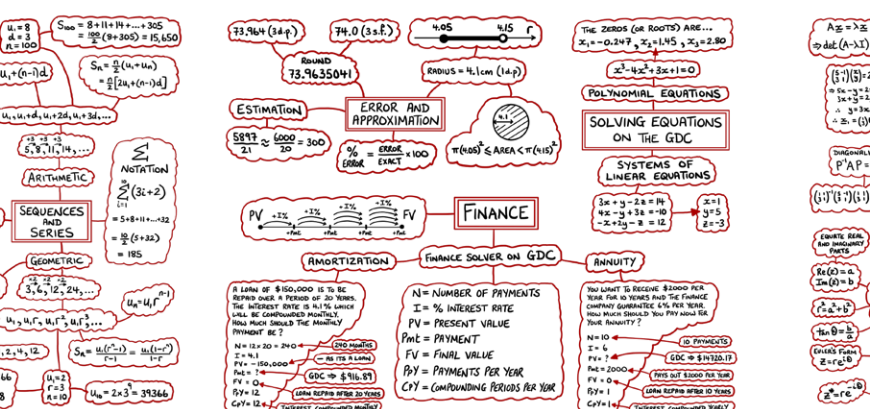
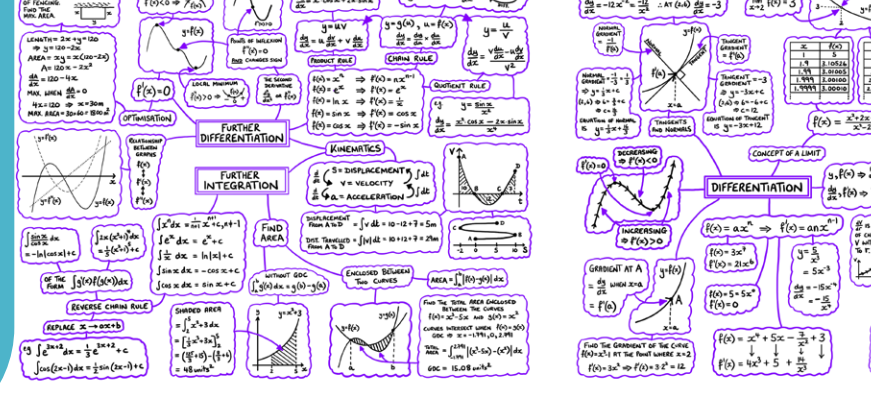
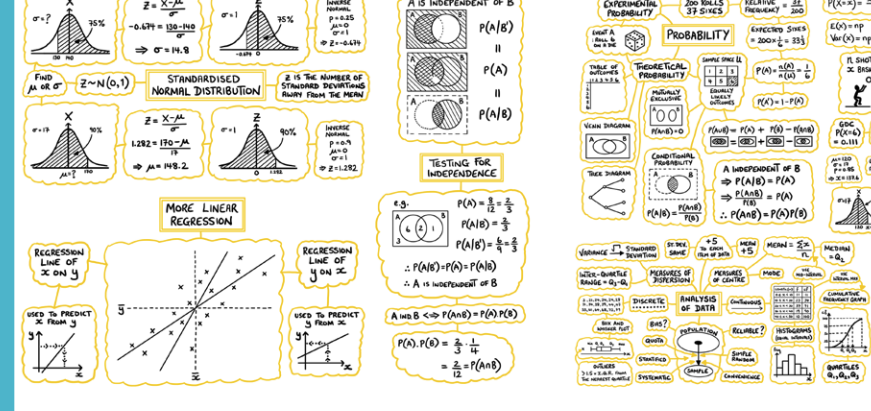
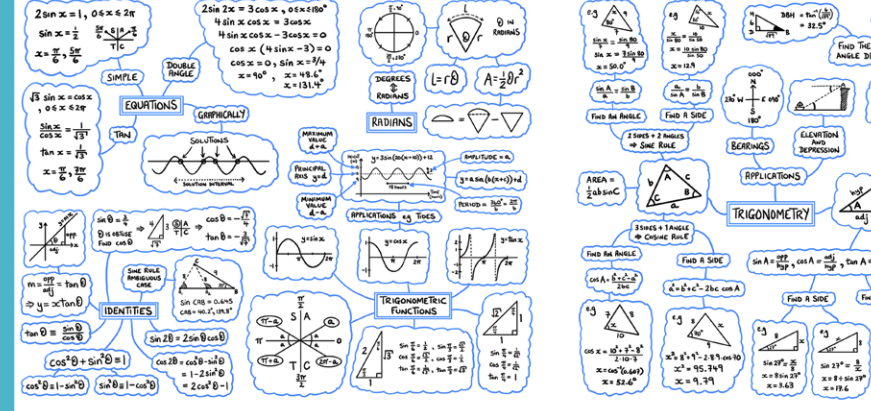
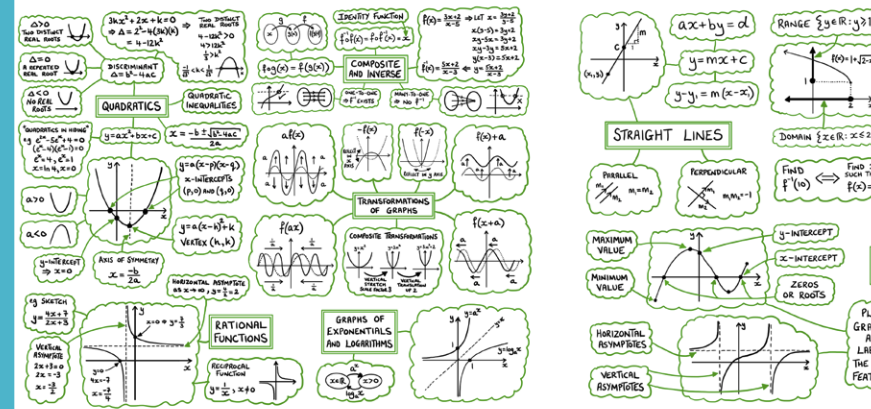
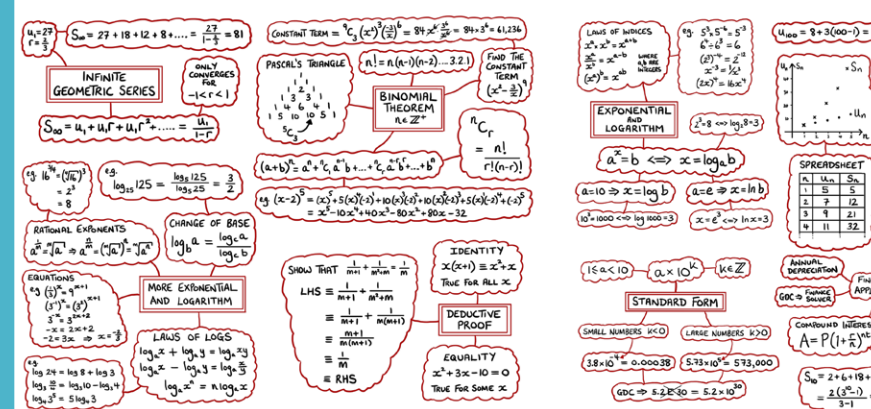
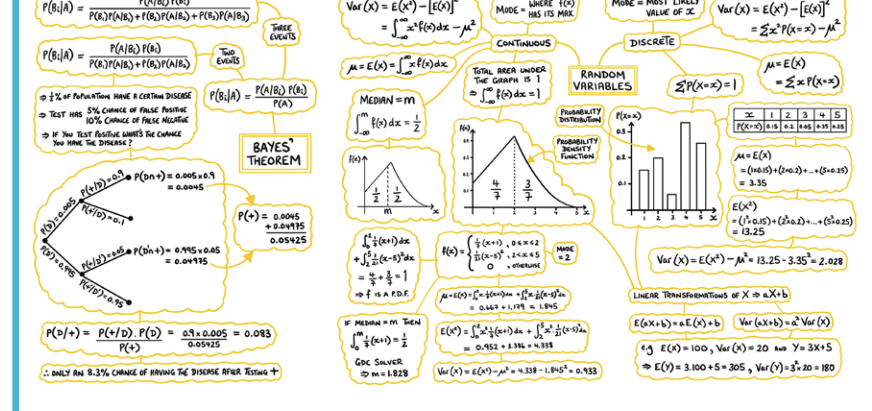
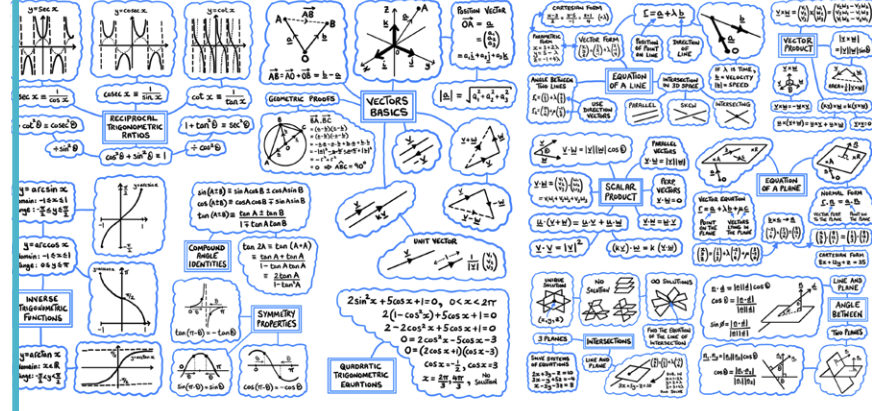
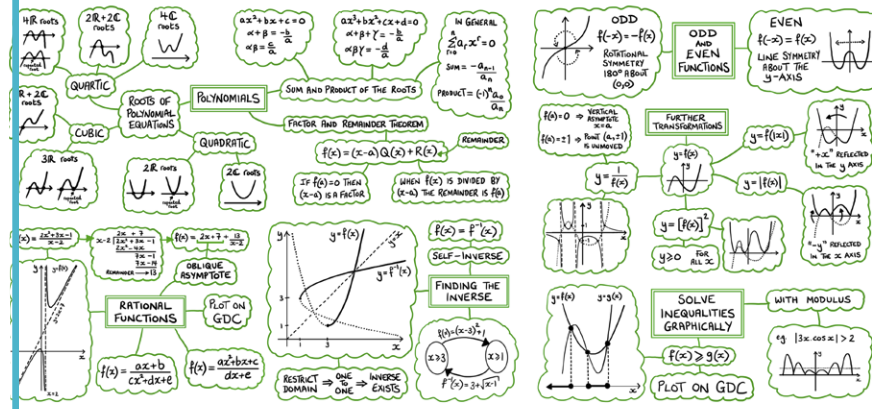
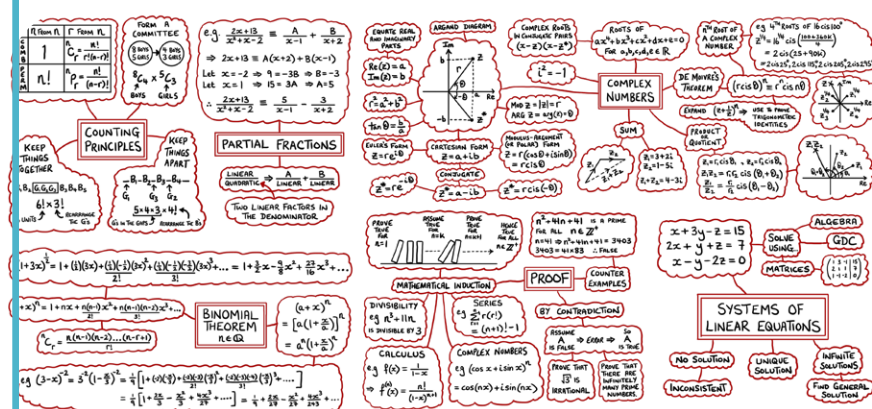
Analysis and approaches AHL

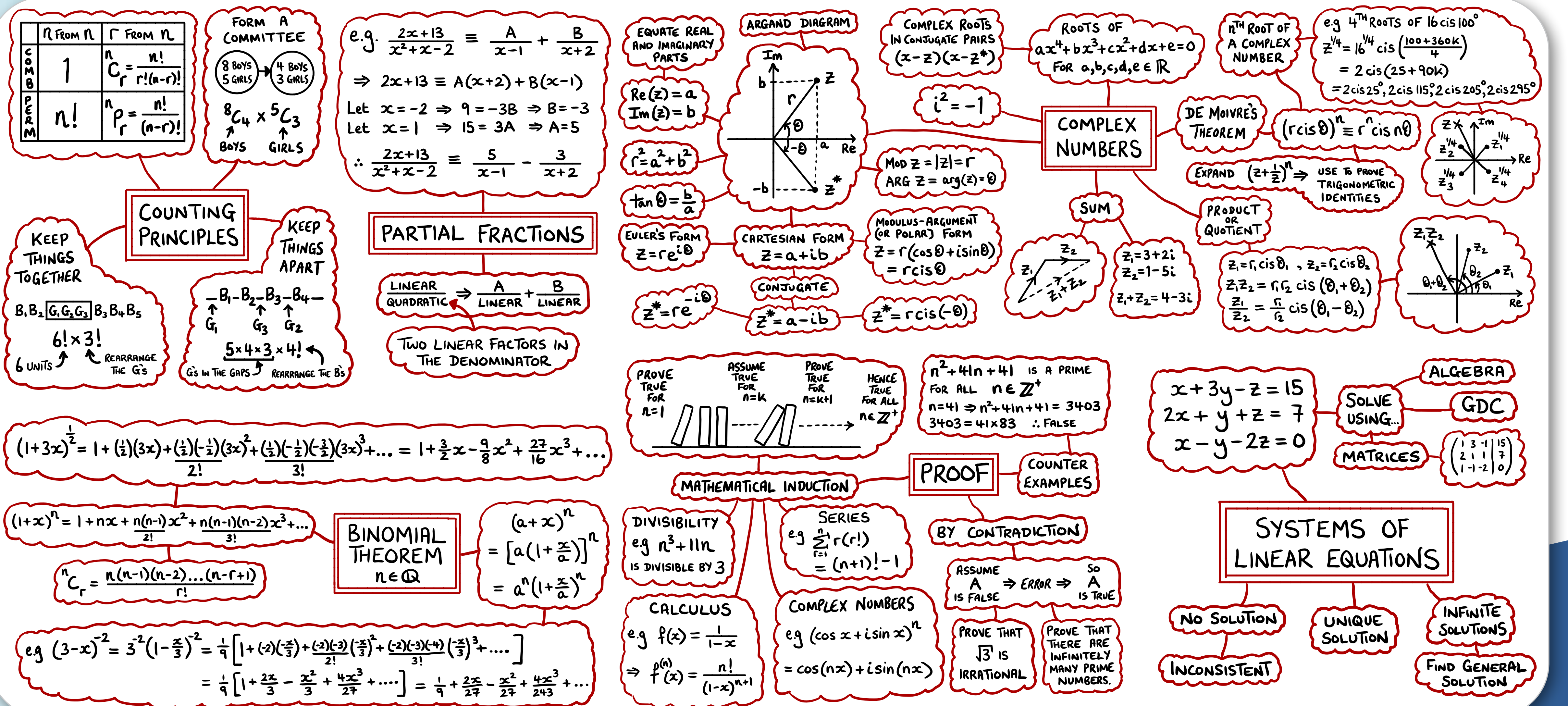
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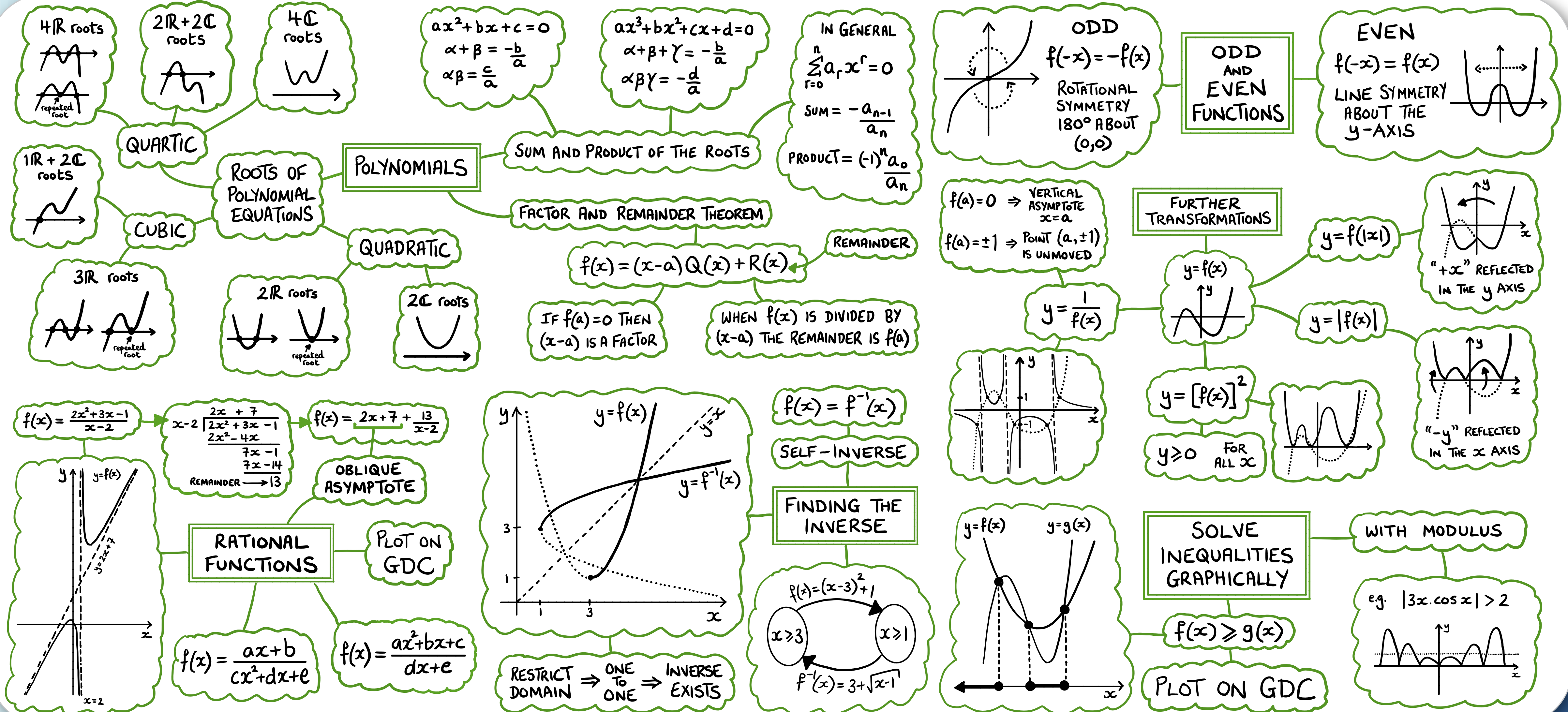
Common content

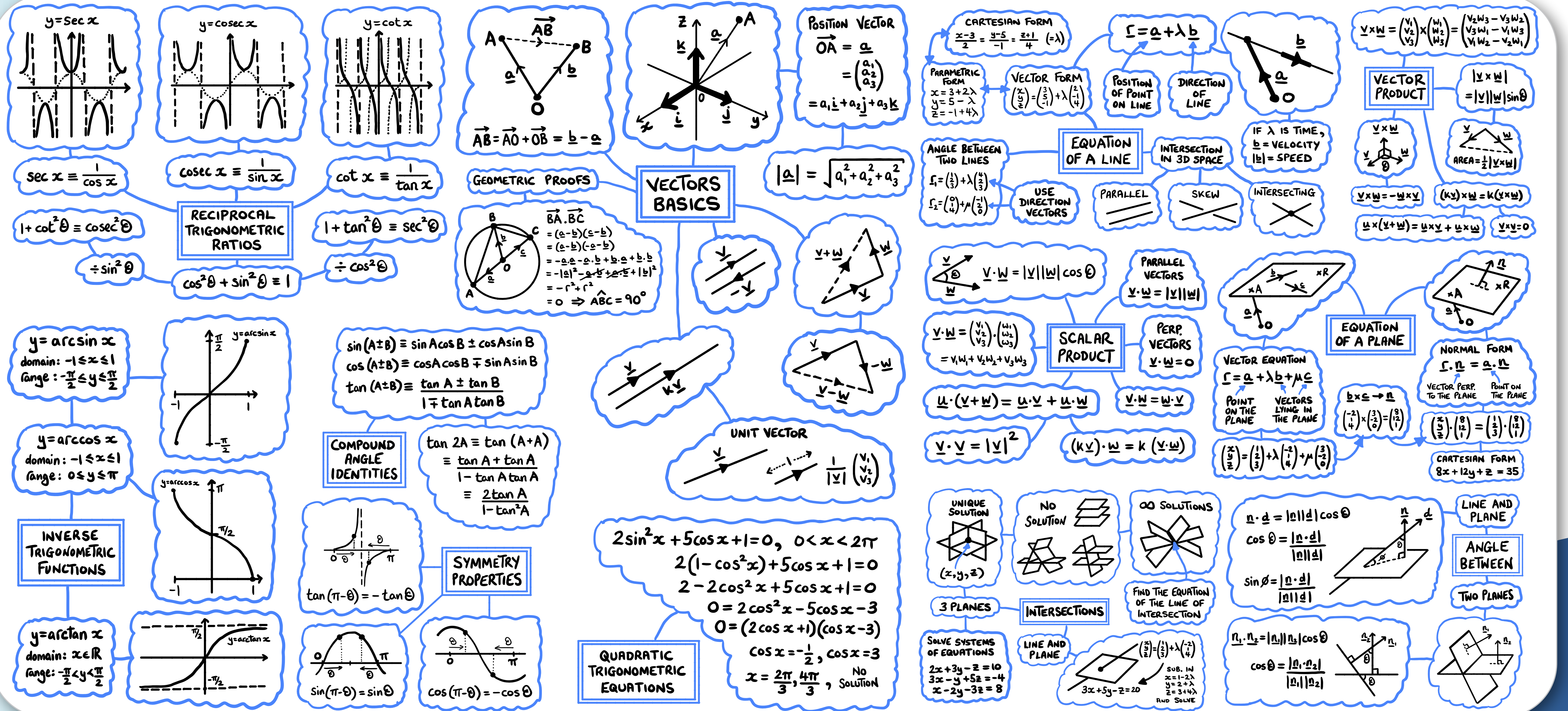
Applications and interpretation SL

Applications and interpretation AHL









BAYES' THEOREM

THREE EVENTS

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)}$$

TWO EVENTS

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}$$

$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}$

⇒ ½% OF POPULATION HAVE A CERTAIN DISEASE
⇒ TEST HAS 5% CHANCE OF FALSE POSITIVE
10% CHANCE OF FALSE NEGATIVE
⇒ IF YOU TEST POSITIVE WHAT'S THE CHANCE YOU HAVE THE DISEASE?

Tree diagram:

- $P(D) = 0.005$
 - $P(+|D) = 0.9$ → $P(D \cap +) = 0.005 \times 0.9 = 0.0045$
 - $P(-|D) = 0.1$
- $P(D') = 0.995$
 - $P(+|D') = 0.05$ → $P(D' \cap +) = 0.995 \times 0.05 = 0.04975$
 - $P(-|D') = 0.95$

$P(+) = \frac{0.0045 + 0.04975}{0.05425}$

$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+)} = \frac{0.9 \times 0.005}{0.05425} = 0.083$

∴ ONLY AN 8.3% CHANCE OF HAVING THE DISEASE AFTER TESTING +

RANDOM VARIABLES

CONTINUOUS

MODE = WHERE $f(x)$ HAS ITS MAX.

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

TOTAL AREA UNDER THE GRAPH IS 1
⇒ $\int_{-\infty}^{\infty} f(x) dx = 1$

DISCRETE

MODE = MOST LIKELY VALUE OF x

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \sum x^2 P(X=x) - \mu^2$$

$$\mu = E(X) = \sum x P(X=x)$$

$\sum P(X=x) = 1$

PROBABILITY DISTRIBUTION

x	1	2	3	4	5
$P(X=x)$	0.15	0.2	0.05	0.35	0.25

$\mu = E(X) = (1 \times 0.15) + (2 \times 0.2) + \dots + (5 \times 0.25) = 3.35$

$E(X^2) = (1^2 \times 0.15) + (2^2 \times 0.2) + \dots + (5^2 \times 0.25) = 13.25$

$\text{Var}(X) = E(X^2) - \mu^2 = 13.25 - 3.35^2 = 2.028$

PROBABILITY DENSITY FUNCTION

Tree diagram for PDF:

- $\int_0^2 \frac{1}{7}(x+1) dx + \int_2^5 \frac{1}{21}(x-5)^2 dx = \frac{4}{7} + \frac{3}{7} = 1$ ⇒ f IS A P.D.F.

$f(x) = \begin{cases} \frac{1}{7}(x+1), & 0 \leq x \leq 2 \\ \frac{1}{21}(x-5)^2, & 2 < x \leq 5 \\ 0, & \text{OTHERWISE} \end{cases}$

MODE = 2

$\mu = E(X) = \int_0^2 x \cdot \frac{1}{7}(x+1) dx + \int_2^5 x \cdot \frac{1}{21}(x-5)^2 dx = 0.667 + 1.179 = 1.845$

$E(X^2) = \int_0^2 x^2 \cdot \frac{1}{7}(x+1) dx + \int_2^5 x^2 \cdot \frac{1}{21}(x-5)^2 dx = 0.952 + 3.386 = 4.338$

$\text{Var}(X) = E(X^2) - \mu^2 = 4.338 - 1.845^2 = 0.933$

IF MEDIAN = m THEN
 $\int_0^m \frac{1}{7}(x+1) dx = \frac{1}{2}$
GDC SOLVER
⇒ $m = 1.828$

LINEAR TRANSFORMATIONS OF $X \Rightarrow aX+b$

$E(aX+b) = aE(X) + b$ $\text{Var}(aX+b) = a^2 \text{Var}(X)$

e.g. $E(X) = 100$, $\text{Var}(X) = 20$ AND $Y = 3X+5$
⇒ $E(Y) = 3 \cdot 100 + 5 = 305$, $\text{Var}(Y) = 3^2 \times 20 = 180$

e.g. $\frac{dy}{dx} = xy - x = x(y-1)$
 $\therefore \int \frac{1}{y-1} dy = \int x dx$
 $\ln|y-1| = \frac{1}{2}x^2 + c$
 $y-1 = e^{\frac{1}{2}x^2 + c}$
 $y-1 = Ae^{\frac{1}{2}x^2}$
 $y = Ae^{\frac{1}{2}x^2} + 1$

SEPARATE AND INTEGRATE
 $\int \frac{1}{g(y)} dy = \int f(x) dx$

VARIABLES SEPARABLE
 $\frac{dy}{dx} = f(x)g(y)$

e.g. $\frac{dy}{dx} = x + y$
 $x_0 = 0, y_0 = 1, h = 0.2$

n	x_n	y_n
0	0	1
1	0.2	$= 1 + 0.2(0 + 1) = 1.2$
2	0.4	$= 1.2 + 0.2(0.2 + 1.2) = 1.48$
3	0.6	$= 1.48 + 0.2(0.4 + 1.48) = 1.856$
4	0.8	$= 1.856 + 0.2(0.6 + 1.856) = 2.3472$
5	1	$= 2.3472 + 0.2(0.8 + 2.3472) = 2.9766$

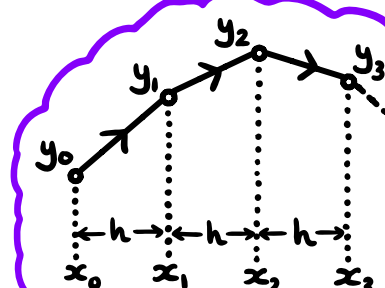
$x_5 = 1, y_5 = 2.9766$

FIRST ORDER EQUATIONS OF THE FORM...

DIFFERENTIAL EQUATIONS

NUMERICAL SOLUTIONS OF $\frac{dy}{dx} = f(x, y)$

EULER'S METHOD WITH CONSTANT STEP h



$y_{n+1} = y_n + hf(x_n, y_n)$
 $x_{n+1} = x_n + h$

e.g. $\frac{dy}{dx} = \frac{x+2y}{x} = 1 + 2(\frac{y}{x})$
 LET $y = vx \Rightarrow \frac{dy}{dx} = x \frac{dv}{dx} + v$
 $\therefore x \frac{dv}{dx} + v = 1 + 2(\frac{vx}{x})$
 $x \frac{dv}{dx} = 1 + v$
 $\int \frac{1}{1+v} dv = \int \frac{1}{x} dx$
 $\ln|1+v| = \ln|x| + c$
 $1 + \frac{y}{x} = Ax$
 $y = Ax^2 - x$

SUBSTITUTE $y = vx$

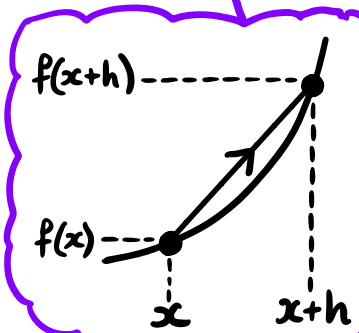
HOMOGENEOUS FORM
 $\frac{dy}{dx} = f(\frac{y}{x})$

e.g. $\frac{dy}{dx} + 2y = e^{3x}$
 $\therefore I(x) = e^{\int 2 dx} = e^{2x}$
 $e^{2x} \frac{dy}{dx} + 2e^{2x}y = e^{3x}$
 $\frac{d}{dx}(y \cdot e^{2x}) = e^{3x}$
 $y \cdot e^{2x} = \int e^{3x} dx$
 $y \cdot e^{2x} = \frac{1}{3}e^{3x} + c$
 $y = \frac{e^x}{3} + \frac{c}{e^{2x}}$

MULTIPLY THROUGH BY $I(x) = e^{\int P(x) dx}$

INTEGRATING FACTOR
 $\frac{dy}{dx} + P(x)y = Q(x)$

$f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$



LIMITS

L'HÔPITAL'S RULE

"0" or "∞"

ONCE
 $\lim_{x \rightarrow 1} \frac{x^3 + x^2 - 2}{3x^2 - 2x - 1} = \lim_{x \rightarrow 1} \frac{3x^2 + 2x}{6x - 2} = \frac{5}{4}$

TWICE
 $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

DIFFERENTIABLE AT $x=a$
 CONTINUOUS CHANGE OF GRADIENT
 SHARP CHANGE OF GRADIENT

CONTINUOUS AT $x=a$ IF
 $\lim_{x \rightarrow a^-} f(x) = f(a) = \lim_{x \rightarrow a^+} f(x)$



$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$

HARD $\int \frac{1}{x} dx$ LET $u = f(x) \rightarrow \int \frac{1}{u} du$

CHANGE LIMITS

$\int x(2x+3)^5 dx$
 LET $u = 2x+3$
 $\frac{du}{dx} = 2, dx = \frac{1}{2} du$
 $x = \frac{1}{2}(u-3)$
 $= \int \frac{1}{2}(u-3)u^5 \cdot \frac{1}{2} du$
 $= \int \frac{1}{4}u^6 - \frac{3}{4}u^5 du$
 $= \frac{1}{28}u^7 - \frac{3}{8}u^6 + c$
 $= \frac{(2x+3)^7}{28} - \frac{(2x+3)^6}{8} + c$

INTEGRATION BY SUBSTITUTION

TRIG. SUBSTITUTIONS

IF YOU SEE THEN USE
 $\sqrt{a^2 - x^2} \Rightarrow x = a \sin \theta$
 $\sqrt{x^2 - a^2} \Rightarrow x = a \sec \theta$
 $\sqrt{a^2 + x^2} \Rightarrow x = a \tan \theta$

$\int_0^{\ln 3} \frac{e^x}{e^{2x} + 9} dx$
 $[u = e^x, \frac{du}{dx} = e^x, du = e^x dx]$
 WHEN $x=0 \Rightarrow u=1$
 $x=\ln 3 \Rightarrow u=3$
 $= \int_1^3 \frac{1}{u^2 + 9} du$
 $= \left[\frac{1}{3} \arctan \frac{u}{3} \right]_1^3$
 $= \frac{1}{3} \arctan 1 - \frac{1}{3} \arctan \frac{1}{3}$
 $= \frac{\pi}{12} - \frac{1}{3} \arctan \frac{1}{3}$

$\int \sec^2 x dx = \tan x + c$
 $\int \sec x \cdot \tan x dx = \sec x + c$
 $\int \csc x \cdot \cot x dx = -\csc x + c$
 $\int \csc^2 x dx = -\cot x + c$

$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
 $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
 $f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$
 $f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$

$f(x) = a^x \Rightarrow f'(x) = a^x \ln a$

$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$

MORE INTEGRALS

MORE DERIVATIVES

$\int a^x dx = \frac{1}{\ln a} a^x + c$

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$

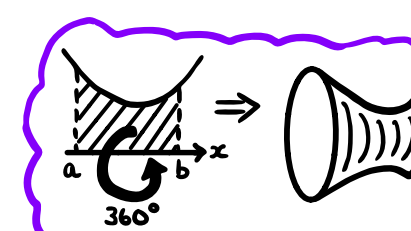
$\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x + c$

$\int \frac{1}{1+x^2} dx = \arctan x + c$

$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

$f(x) = \arccos x \Rightarrow f'(x) = \frac{-1}{\sqrt{1-x^2}}$

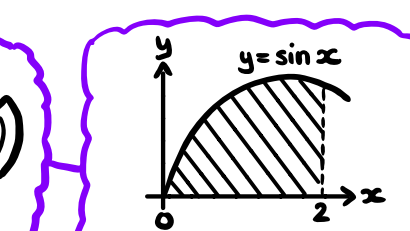
$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$



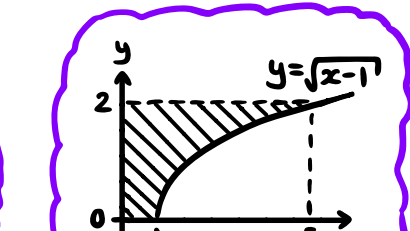
360° ABOUT THE x-AXIS
 VOLUME = $\int_a^b \pi y^2 dx$

VOLUME OF REVOLUTION

360° ABOUT THE y-AXIS
 VOLUME = $\int_a^b \pi x^2 dy$



ROTATE THE SHADED AREA 360° ABOUT THE x-AXIS
 $\therefore V = \pi \int_0^2 \sin^2 x dx$
 $= 3.74 \text{ units}^3$



ROTATE THE SHADED AREA 360° ABOUT THE y-AXIS
 $y = \sqrt{x-1} \Rightarrow y^2 + 1 = x$
 $\therefore V = \int_0^2 \pi (y^2 + 1)^2 dy$
 $= 43.14 \text{ units}^3$

CHAIN RULE

FIND GRADIENT AT (a,b) $\Rightarrow$ SUB. IN VALUES FIRST

PRODUCT RULE

IMPLICIT DIFFERENTIATION

$y^5 \Rightarrow 5y^4 \frac{dy}{dx}$

$x^3 y^2 \Rightarrow 3x^2 y \frac{dy}{dx} + 2xy^2 \frac{dx}{dx}$

$e^{2x+y} + 2xy^3 = 7$
 $e^{2x+y} (2 + \frac{dy}{dx}) + 2x \cdot 3y^2 \frac{dy}{dx} + 2y^3 = 0$
 $\frac{dy}{dx} = \frac{-2y^3 - e^{2x+y}}{e^{2x+y} + 6xy^2}$

$V = \frac{1}{3} \pi r^2 h$

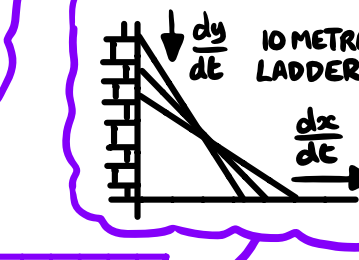
$\frac{dV}{dt} = \frac{1}{3} \pi (r^2 \frac{dh}{dt} + 2r \frac{dr}{dt} h)$



RELATED RATES

$x^2 + y^2 = 100$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$



$f(0) = \cos(0) = 1$
 $f'(0) = -\sin(0) = 0$
 $f''(0) = -\cos(0) = -1$
 $f'''(0) = \sin(0) = 0$
 $f^{(4)}(0) = \cos(0) = 1$
 etc...

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$
 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

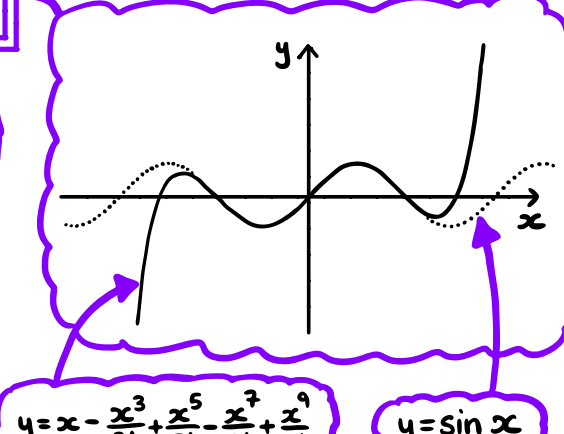
MACLAURIN SERIES

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
 $x \rightarrow x^2 \Rightarrow e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$

FROM A DIFFERENTIAL EQUATION

$y' = x^2 + y^2, x=0, y=1$
 DIFFERENTIATE IMPLICITLY
 $\Rightarrow y'' = 2x + 2yy'$
 $\Rightarrow y''' = 2 + 2yy'' + 2y'y'$

$y(0) = 1$
 $y'(0) = 0^2 + 1^2 = 1$
 $y''(0) = 2 \cdot 0 + 2 \cdot 1 \cdot 1 = 2$
 $y'''(0) = 2 + 2 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 1 = 8$
 $y = 1 + 1 \cdot x + \frac{2 \cdot x^2}{2!} + \frac{8 \cdot x^3}{3!} + \dots$



$\int \ln x dx$
 $= \int \ln x \cdot 1 dx$
 $= \ln x \cdot x - \int x \cdot \frac{1}{x} dx$
 $= x \ln x - x + c$

MULTIPLY BY 1

LOOP ROUND

$I = \int e^x \sin x dx$
 $= e^x \cdot \cos x + \int e^x \cos x dx$
 $= e^x \cos x + [e^x \sin x - \int e^x \sin x dx]$
 $= e^x \cos x + e^x \sin x - I$
 $\therefore 2I = e^x \sin x - e^x \cos x$
 $I = \frac{1}{2}(e^x \sin x - e^x \cos x) + c$

$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx} dx$

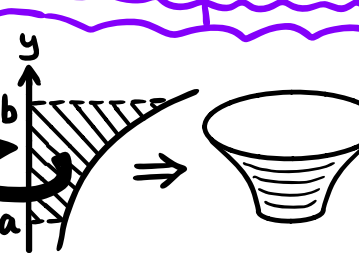
INTEGRATION BY PARTS

PARTS TWICE

$\int x^2 \sin x dx$
 $= x^2 \cdot \cos x + \int 2x \cos x dx$
 $= -x^2 \cos x + [2x \sin x - \int 2 \sin x dx]$
 $= -x^2 \cos x + 2x \sin x + 2 \cos x + c$

PARTS ONCE

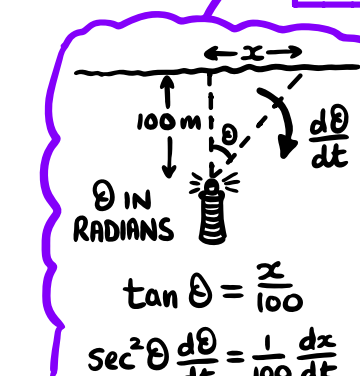
$\int x e^x dx$
 $= x e^x - \int e^x \cdot 1 dx$
 $= x e^x - e^x + c$



INTEGRATION WITH PARTIAL FRACTIONS

PARTIAL FRACTIONS
 $\Rightarrow \frac{1}{x^2 + 3x + 2} = \frac{1}{(x+1)(x+2)}$
 $= \frac{1}{x+1} - \frac{1}{x+2}$

$\int \frac{1}{x^2 + 3x + 2} dx = \int \frac{1}{x+1} - \frac{1}{x+2} dx$
 $= \ln|x+1| - \ln|x+2| = \ln \left| \frac{x+1}{x+2} \right| + c$



$\tan \theta = \frac{y}{x}$
 $\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{100} \frac{dx}{dt}$

$D^2 = x^2 + y^2$
 $2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$

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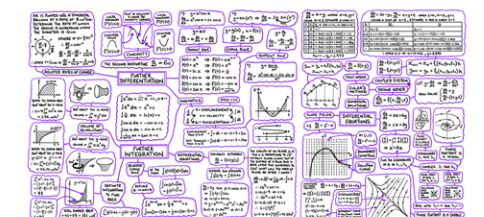
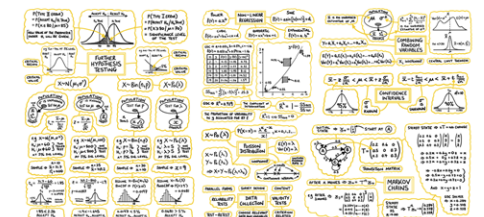
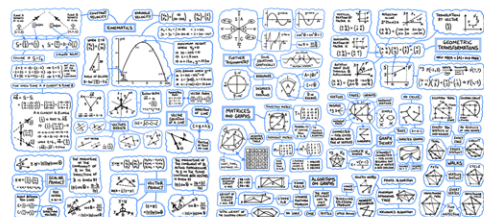
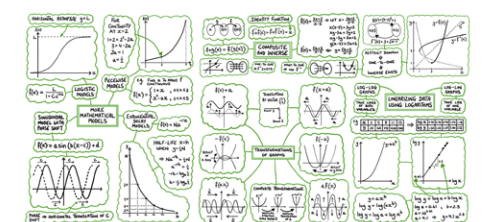
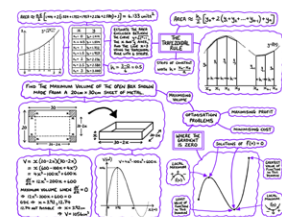
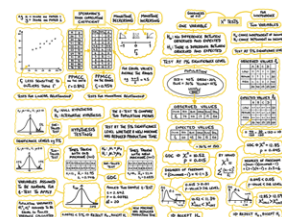
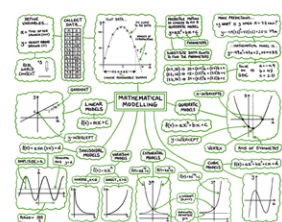
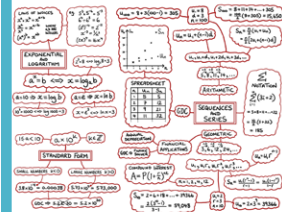
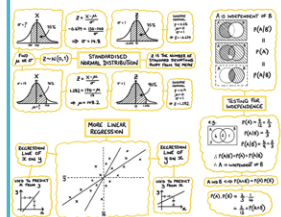
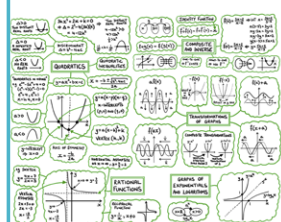
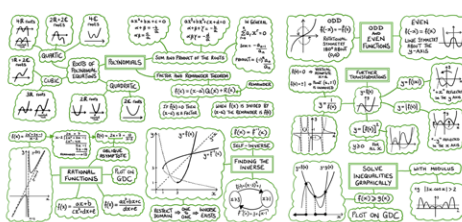
Number and algebra

Functions

Geometry and trigonometry

Statistics and probability

Calculus



$$u_1 = 27$$

$$r = \frac{2}{3}$$

$$S_{\infty} = 27 + 18 + 12 + 8 + \dots = \frac{27}{1 - \frac{2}{3}} = 81$$

INFINITE GEOMETRIC SERIES

ONLY CONVERGES FOR
 $-1 < r < 1$

$$S_{\infty} = u_1 + u_1 r + u_1 r^2 + \dots = \frac{u_1}{1-r}$$

e.g. $16^{\frac{3}{4}} = (\sqrt[4]{16})^3$
 $= 2^3$
 $= 8$

e.g. $\log_{25} 125 = \frac{\log_5 125}{\log_5 25} = \frac{3}{2}$

RATIONAL EXPONENTS

$$a^{\frac{1}{m}} = \sqrt[m]{a} \Rightarrow a^{\frac{n}{m}} = (\sqrt[m]{a})^n = \sqrt[m]{a^n}$$

CHANGE OF BASE

$$\log_b a = \frac{\log_c a}{\log_c b}$$

EQUATIONS

e.g. $\left(\frac{1}{3}\right)^x = 9^{x+1}$
 $(3^{-1})^x = (3^2)^{x+1}$
 $3^{-x} = 3^{2x+2}$
 $-x = 2x+2$
 $-2 = 3x \Rightarrow x = -\frac{2}{3}$

MORE EXPONENTIAL AND LOGARITHM

e.g. $\log 24 = \log 8 + \log 3$
 $\log_3 \frac{10}{4} = \log_3 10 - \log_3 4$
 $\log_4 3^5 = 5 \log_4 3$

LAWS OF LOGS

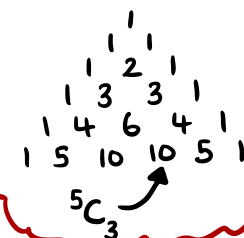
$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

$$\text{CONSTANT TERM} = {}^9C_3 (x^2)^3 \left(\frac{-3}{x}\right)^6 = 84 \cancel{x^6} \cdot \frac{3^6}{\cancel{x^6}} = 84 \times 3^6 = 61,236$$

PASCAL'S TRIANGLE



$$n! = n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1$$

BINOMIAL THEOREM

$n \in \mathbb{Z}^+$

FIND THE CONSTANT TERM
 $(x^2 - \frac{3}{x})^9$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \dots + {}^nC_r a^{n-r} b^r + \dots + b^n$$

e.g. $(x-2)^5 = (x)^5 + 5(x)^4(-2)^1 + 10(x)^3(-2)^2 + 10(x)^2(-2)^3 + 5(x)^1(-2)^4 + (-2)^5$
 $= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

SHOW THAT $\frac{1}{m+1} + \frac{1}{m^2+m} = \frac{1}{m}$

$$\begin{aligned} \text{LHS} &\equiv \frac{1}{m+1} + \frac{1}{m^2+m} \\ &\equiv \frac{1}{m+1} + \frac{1}{m(m+1)} \\ &\equiv \frac{m+1}{m(m+1)} \\ &\equiv \frac{1}{m} \\ &\equiv \text{RHS} \end{aligned}$$

IDENTITY

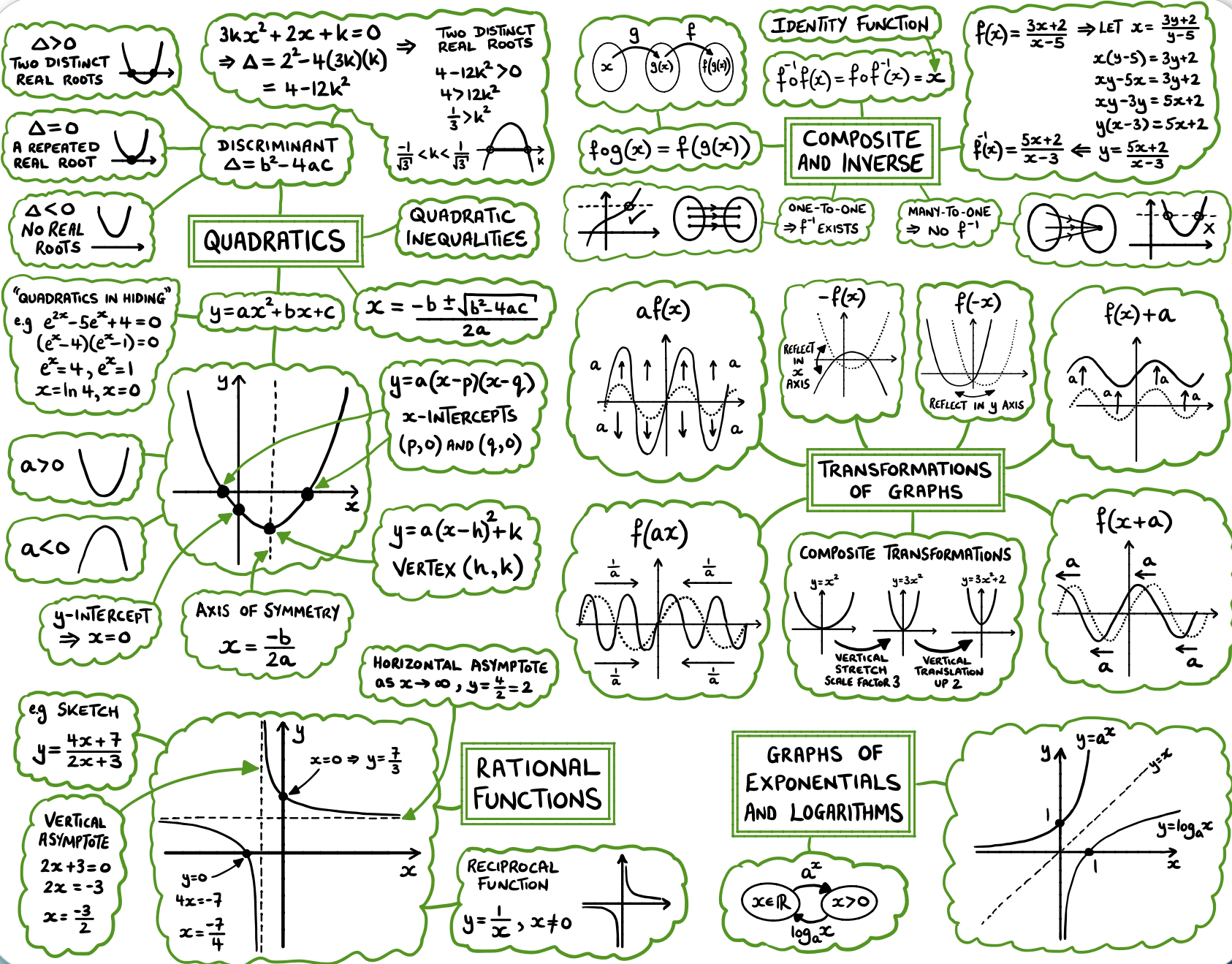
$$x(x+1) \equiv x^2 + x$$

TRUE FOR ALL x

DEDUCTIVE PROOF

EQUALITY
 $x^2 + 3x - 10 = 0$
TRUE FOR SOME x

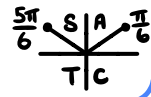
BACK
TO
MAIN



$$2 \sin x = 1, 0 \leq x \leq 2\pi$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$



SIMPLE

DOUBLE
ANGLE

$$2 \sin 2x = 3 \cos x, 0 \leq x \leq 180^\circ$$

$$4 \sin x \cos x = 3 \cos x$$

$$4 \sin x \cos x - 3 \cos x = 0$$

$$\cos x (4 \sin x - 3) = 0$$

$$\cos x = 0, \sin x = \frac{3}{4}$$

$$x = 90^\circ, x = 48.6^\circ$$

$$x = 131.4^\circ$$

EQUATIONS

GRAPHICALLY

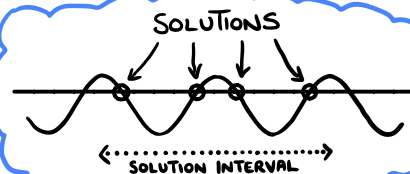
$$\sqrt{3} \sin x = \cos x, 0 \leq x \leq 2\pi$$

$$\frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}}$$

$$\tan x = \frac{1}{\sqrt{3}}$$

$$x = \frac{\pi}{6}, \frac{7\pi}{6}$$

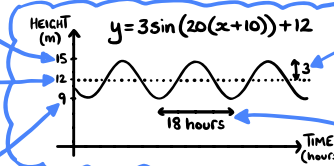
TAN



MAXIMUM
VALUE
 $d+a$

PRINCIPAL
AXIS
 $y=d$

MINIMUM
VALUE
 $d-a$

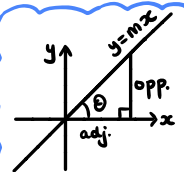


AMPLITUDE = a

$$y = a \sin(b(x+c)) + d$$

$$\text{PERIOD} = \frac{360^\circ}{b} = \frac{2\pi}{b}$$

APPLICATIONS e.g. TIDES



$$\sin \theta = \frac{3}{4} \Rightarrow \frac{4}{3} \text{ (S/A)} \Rightarrow \cos \theta = -\frac{\sqrt{7}}{4}$$

$$\tan \theta = -\frac{3}{\sqrt{7}}$$

θ IS OBTUSE
FIND $\cos \theta$

SINE RULE
AMBIGUOUS
CASE



$$\sin CAB = 0.645$$

$$CAB = 40.2^\circ, 139.8^\circ$$

IDENTITIES

$$m = \frac{\text{opp}}{\text{adj}} = \tan \theta$$

$$\Rightarrow y = x \tan \theta$$

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$$

$$\cos^2 \theta + \sin^2 \theta \equiv 1$$

$$\cos^2 \theta \equiv 1 - \sin^2 \theta$$

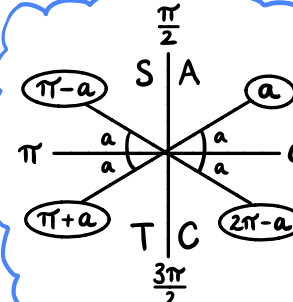
$$\sin^2 \theta \equiv 1 - \cos^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= 1 - 2 \sin^2 \theta$$

$$= 2 \cos^2 \theta - 1$$



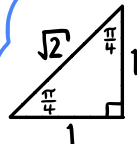
TRIGONOMETRIC
FUNCTIONS



$$\sin \frac{\pi}{6} = \frac{1}{2}, \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}, \tan \frac{\pi}{3} = \sqrt{3}$$

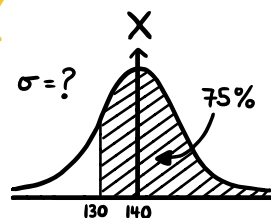


$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan \frac{\pi}{4} = 1$$

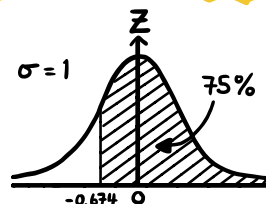
BACK
TO
MAIN



$$Z = \frac{X - \mu}{\sigma}$$

$$-0.674 = \frac{130 - 140}{\sigma}$$

$$\Rightarrow \sigma = 14.8$$



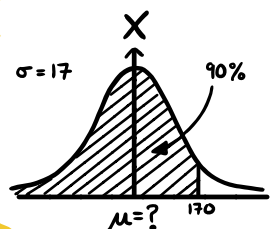
INVERSE
NORMAL
 $p = 0.25$
 $\mu = 0$
 $\sigma = 1$
 $\Rightarrow Z = -0.674$

FIND
 μ OR σ

$Z \sim N(0, 1)$

STANDARDISED
NORMAL DISTRIBUTION

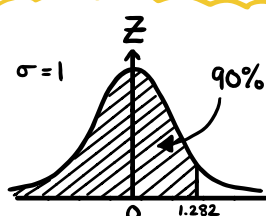
Z IS THE NUMBER OF
STANDARD DEVIATIONS
AWAY FROM THE MEAN



$$Z = \frac{X - \mu}{\sigma}$$

$$1.282 = \frac{170 - \mu}{17}$$

$$\Rightarrow \mu = 148.2$$

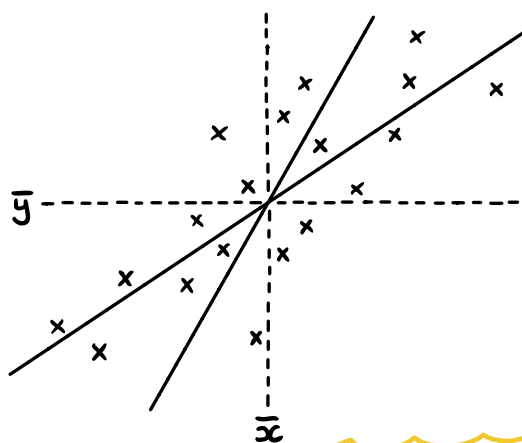
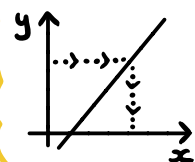


INVERSE
NORMAL
 $p = 0.9$
 $\mu = 0$
 $\sigma = 1$
 $\Rightarrow Z = 1.282$

MORE LINEAR
REGRESSION

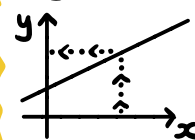
REGRESSION
LINE OF
X ON Y

USED TO PREDICT
X FROM Y

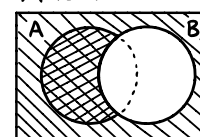


REGRESSION
LINE OF
Y ON X

USED TO PREDICT
Y FROM X

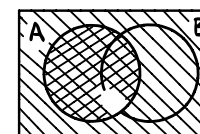


A IS INDEPENDENT OF B



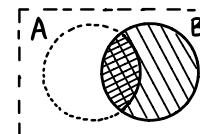
$P(A/B')$

$\parallel$



$P(A)$

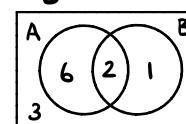
$\parallel$



$P(A/B)$

TESTING FOR
INDEPENDENCE

e.g.



$$P(A) = \frac{8}{12} = \frac{2}{3}$$

$$P(A/B) = \frac{2}{3}$$

$$P(A/B') = \frac{6}{9} = \frac{2}{3}$$

$$\therefore P(A/B') = P(A) = P(A/B)$$

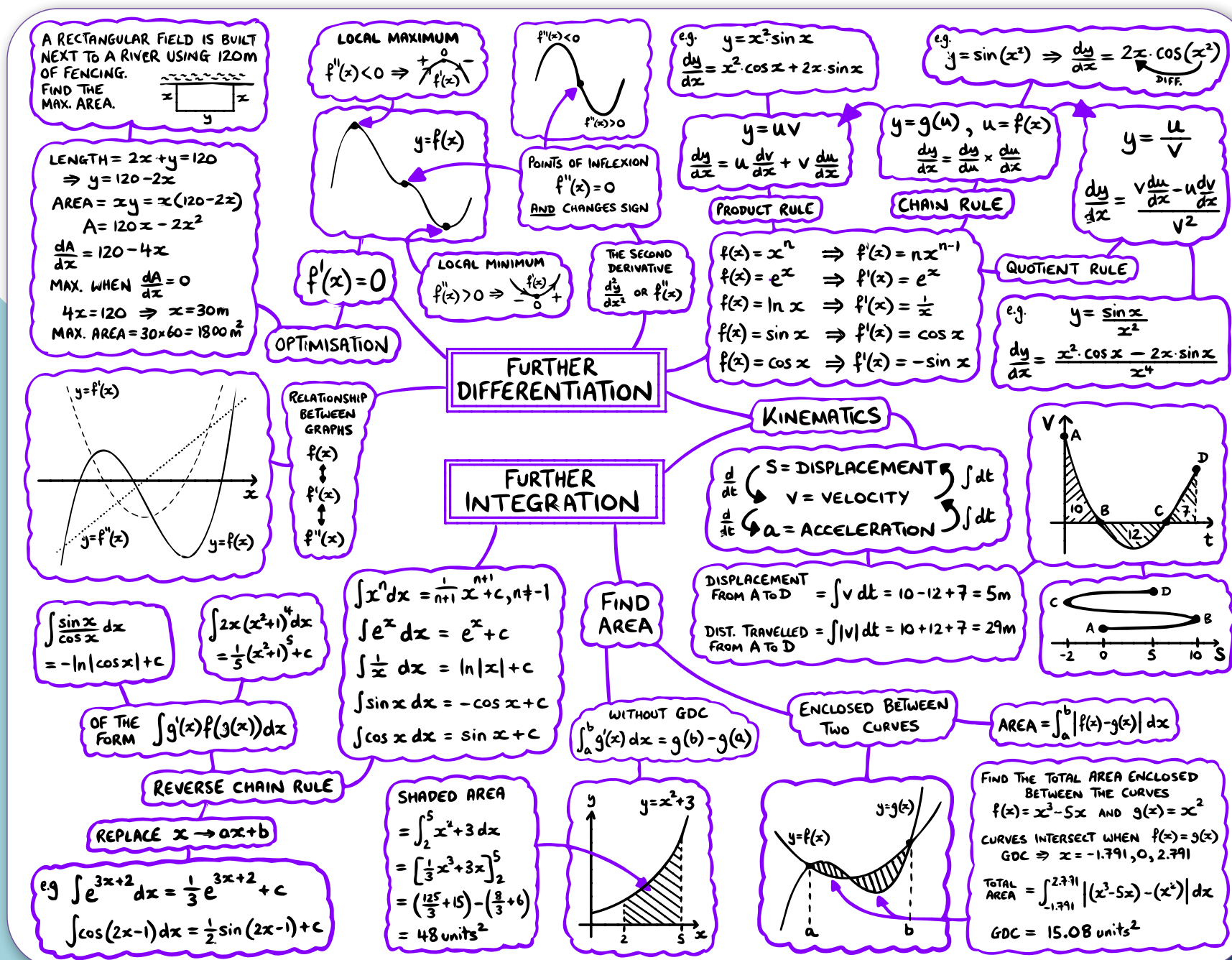
$\therefore A$ IS INDEPENDENT OF B

$$A \text{ IND. } B \Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$$

$$P(A) \cdot P(B) = \frac{2}{3} \cdot \frac{1}{4}$$

$$= \frac{2}{12} = P(A \cap B)$$

BACK
TO
MAIN



Analysis and approaches AHL

Analysis and approaches SL

Common content

Applications and interpretation SL

Applications and interpretation AHL

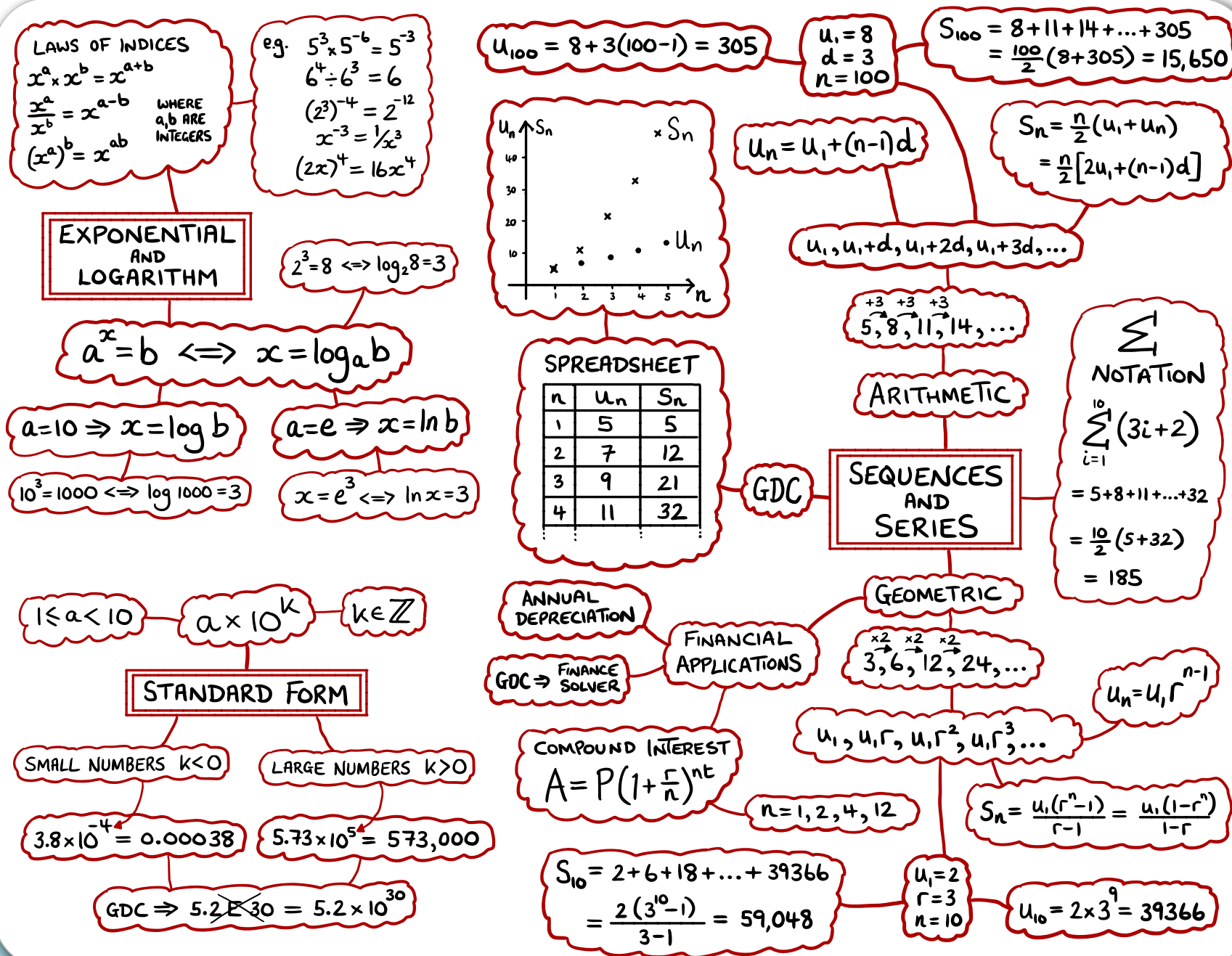
Number and algebra

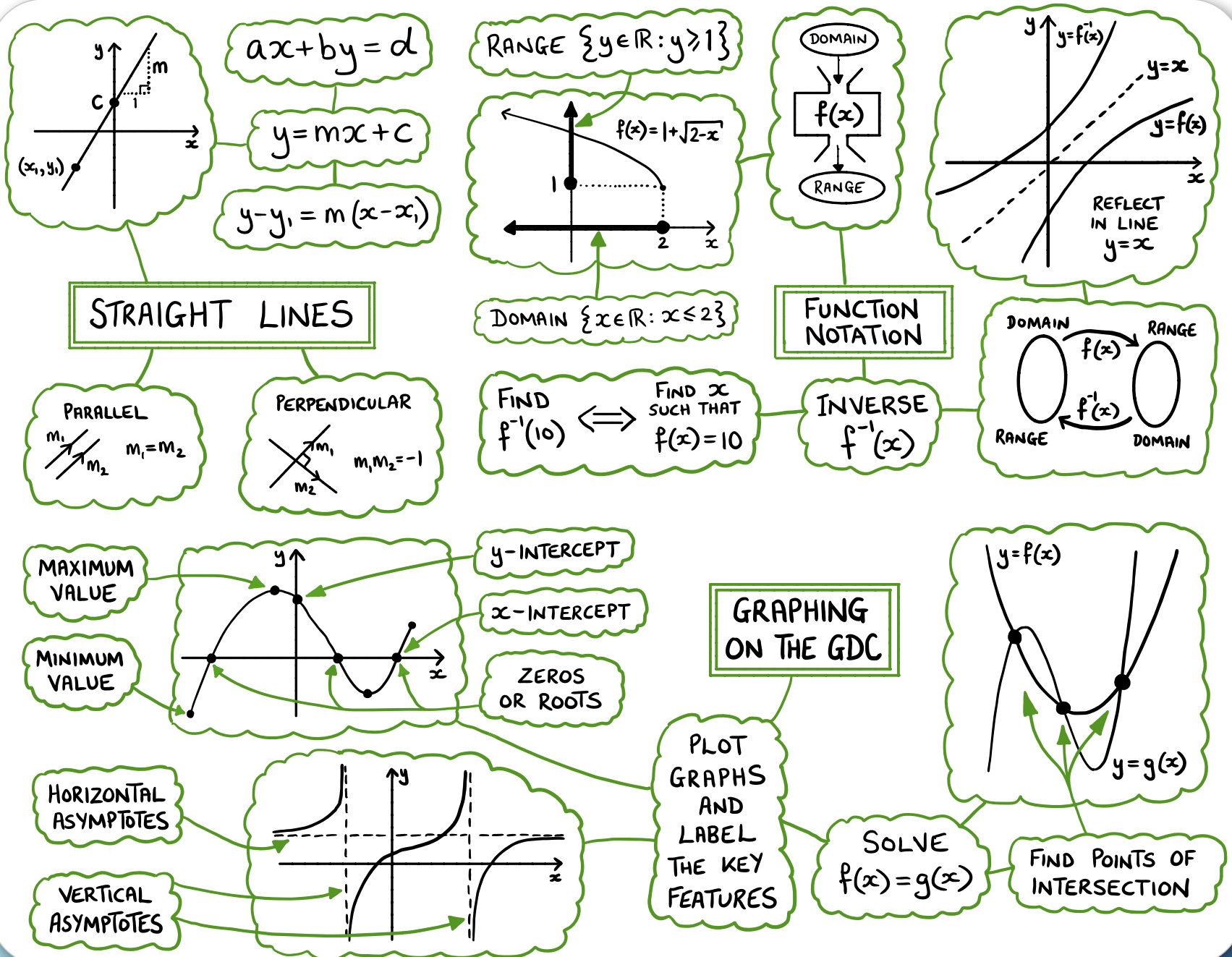
Functions

Geometry and trigonometry

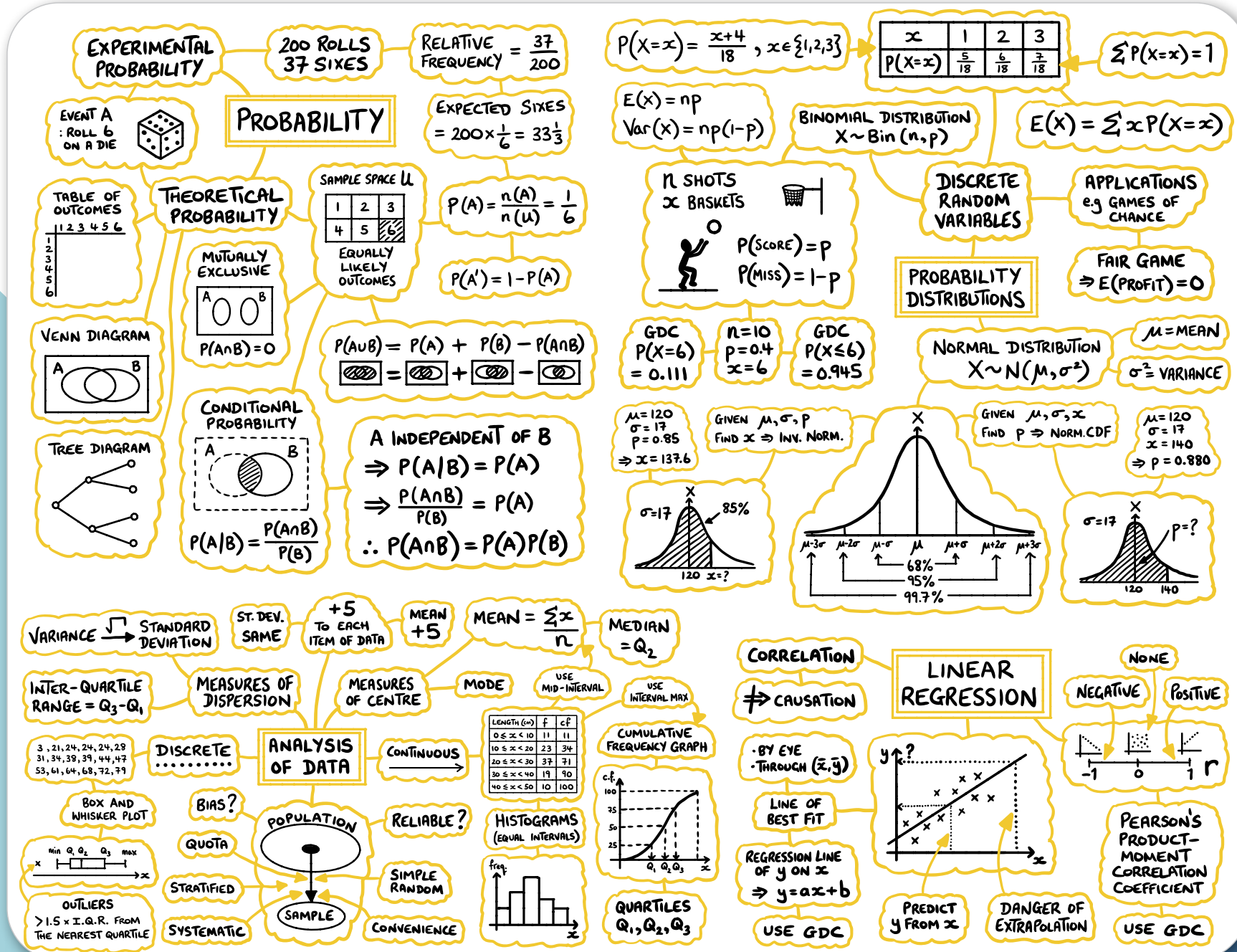
Statistics and probability

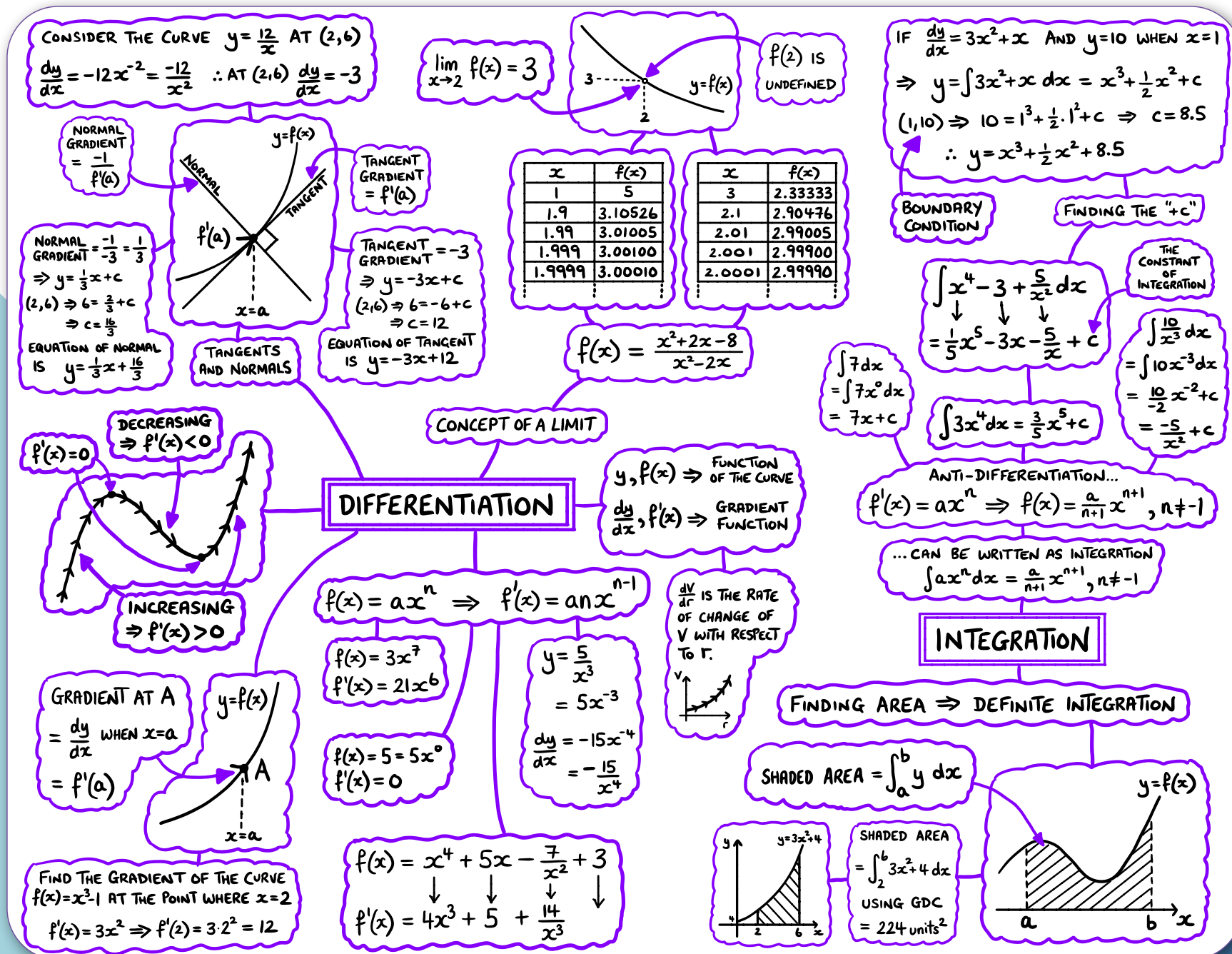
Calculus











Mathematics mind map

Number and algebra

Functions

Geometry and trigonometry

Statistics and probability

Calculus

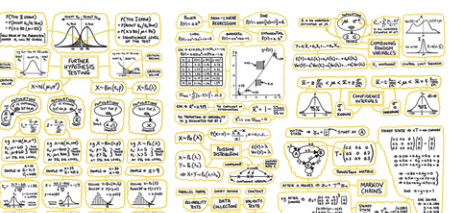
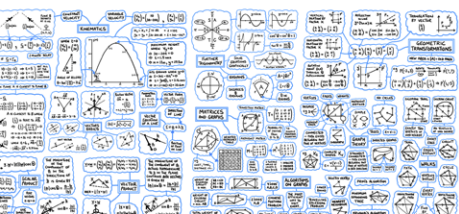
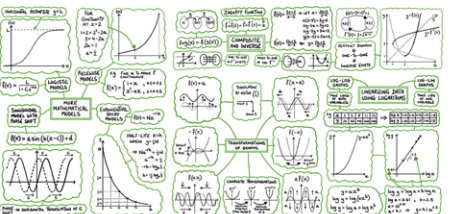
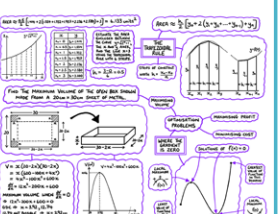
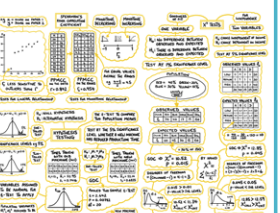
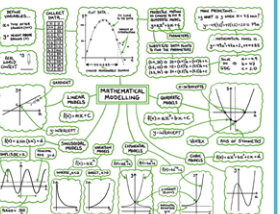
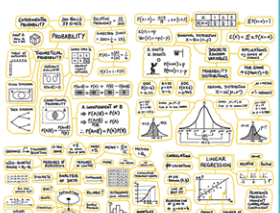
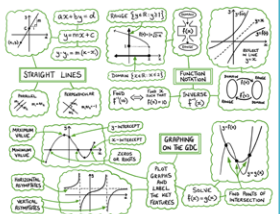
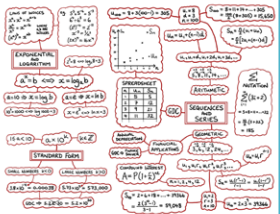
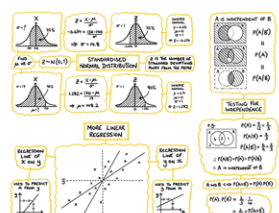
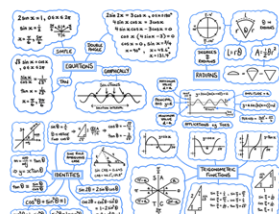
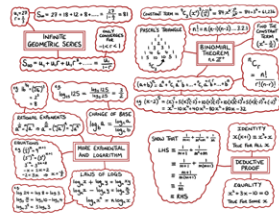
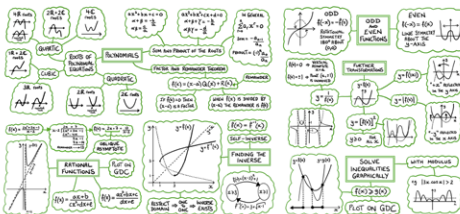
Analysis and approaches AHL

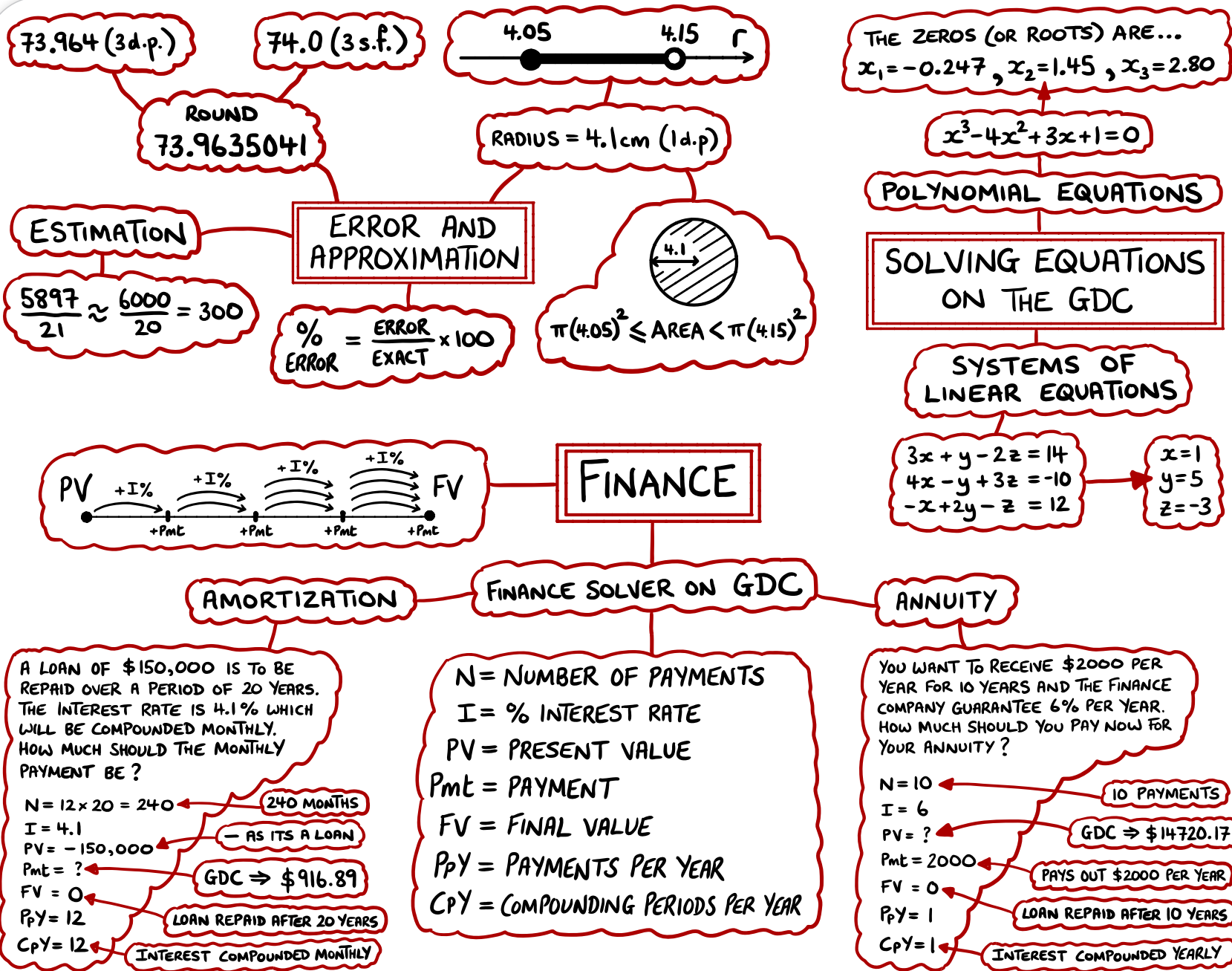
Analysis and approaches SL

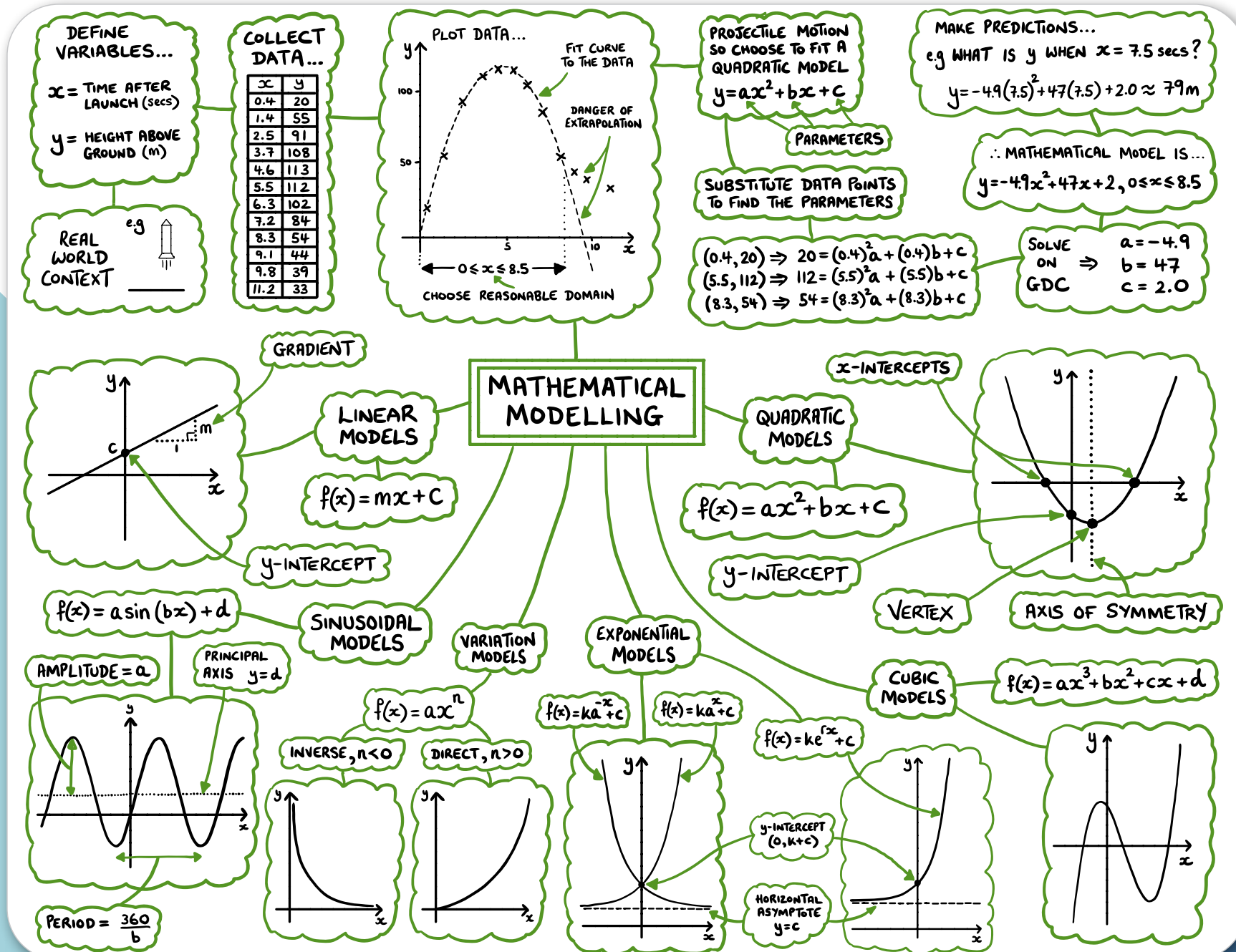
Common content

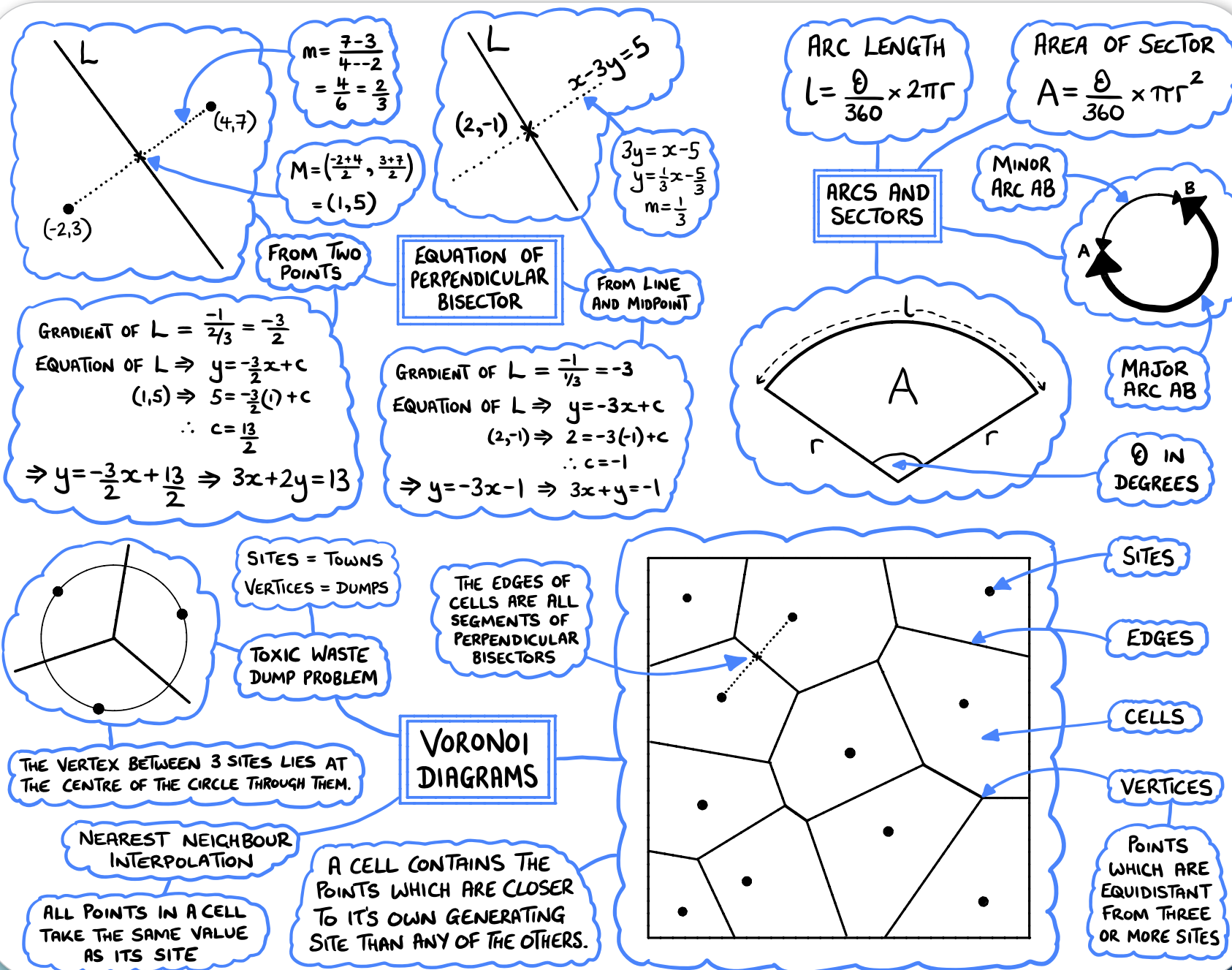
Applications and interpretation SL

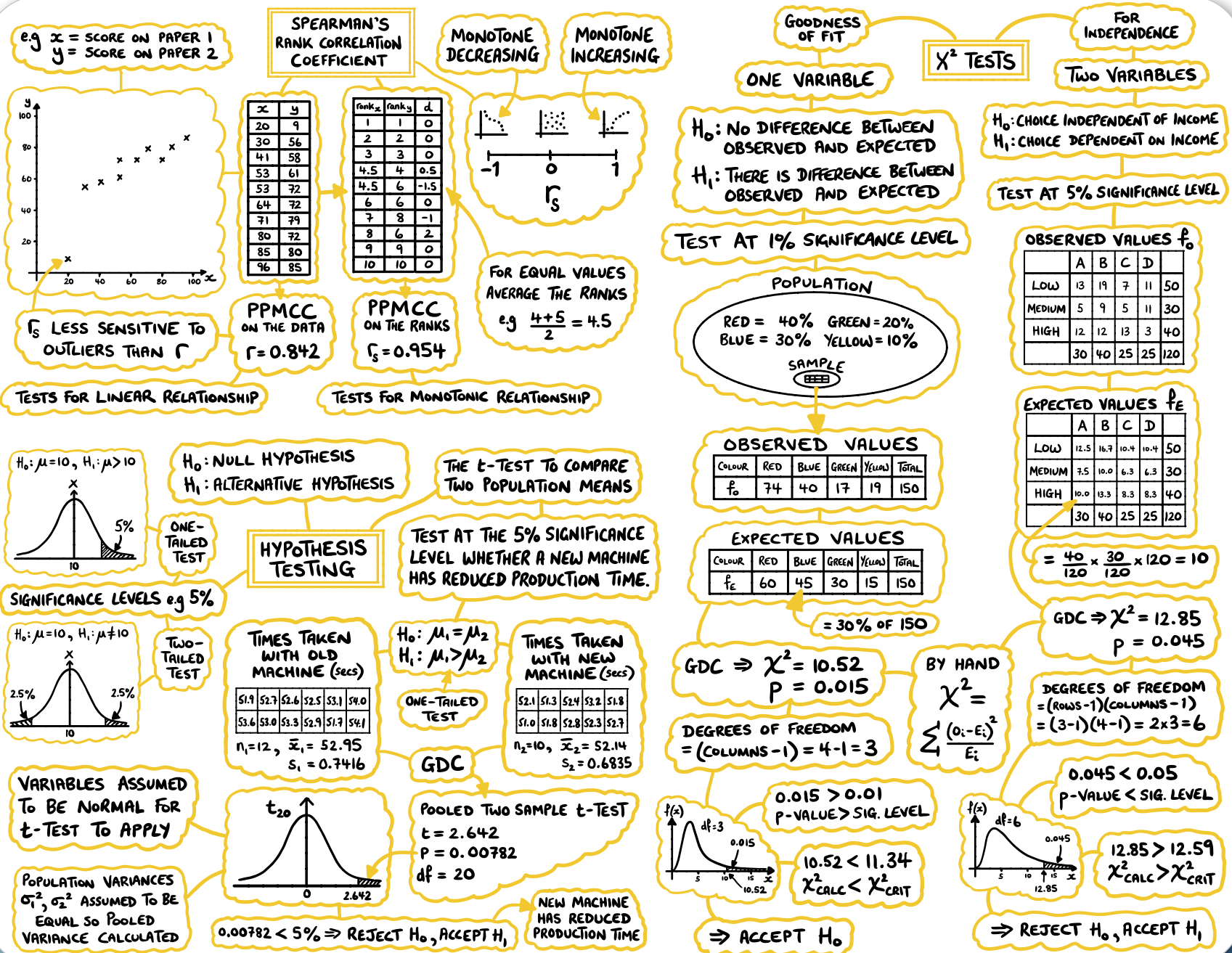
Applications and interpretation AHL



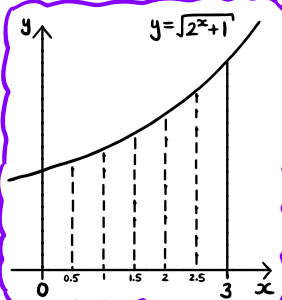








$$\text{AREA} \approx \frac{0.5}{2} [1.414 + 2(1.554 + 1.732 + 1.957 + 2.236 + 2.580) + 3] = 6.133 \text{ units}^2$$



x	y
$x_0 = 0$	$y_0 = 1.414$
$x_1 = 0.5$	$y_1 = 1.554$
$x_2 = 1$	$y_2 = 1.732$
$x_3 = 1.5$	$y_3 = 1.957$
$x_4 = 2$	$y_4 = 2.236$
$x_5 = 2.5$	$y_5 = 2.580$
$x_6 = 3$	$y_6 = 3.000$

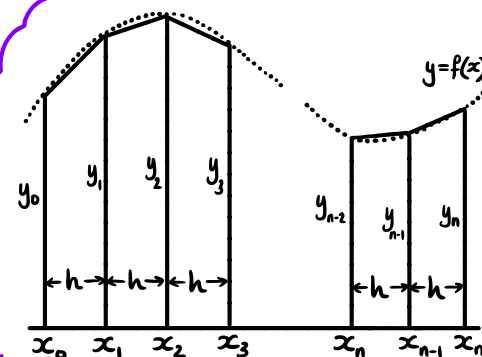
ESTIMATE THE AREA ENCLOSED BETWEEN THE CURVE $y = \sqrt{2x+1}$, THE x AND y AXES, AND THE LINE $x=3$ USING THE TRAPEZOIDAL RULE WITH 6 STRIPS.

$$h = \frac{3-0}{6} = 0.5$$

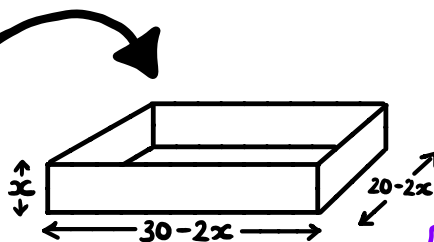
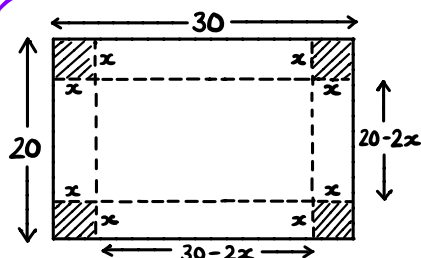
$$\text{AREA} \approx \frac{h}{2} [y_0 + 2(y_1 + y_2 + \dots + y_{n-1}) + y_n]$$

THE TRAPEZOIDAL RULE

STRIPS OF CONSTANT WIDTH $h = \frac{x_n - x_0}{n}$



FIND THE MAXIMUM VOLUME OF THE OPEN BOX SHOWN MADE FROM A 20cm x 30cm SHEET OF METAL.



MAXIMISING VOLUME

OPTIMISATION PROBLEMS

MAXIMISING PROFIT

MINIMISING COST

WHERE THE GRADIENT IS ZERO

SOLUTIONS OF $f'(x) = 0$

$$\begin{aligned} V &= x(20-2x)(30-2x) \\ &= x(600 - 100x + 4x^2) \\ &= 4x^3 - 100x^2 + 600x \end{aligned}$$

$$\frac{dV}{dx} = 12x^2 - 200x + 600$$

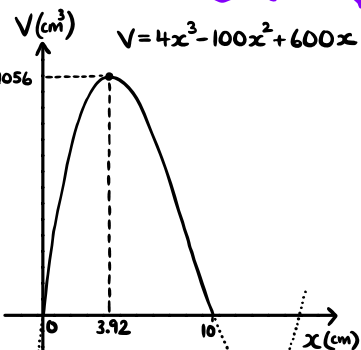
MAXIMUM VOLUME WHEN $\frac{dV}{dx} = 0$

$$\Rightarrow 12x^2 - 200x + 600 = 0$$

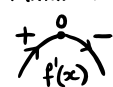
$$\text{GDC} \Rightarrow x = 3.92, 12.74$$

$$12.74 \text{ NOT POSSIBLE} \Rightarrow x = 3.92 \text{ cm}$$

$$\Rightarrow V = 1056 \text{ cm}^3$$



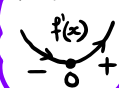
LOCAL MAXIMUM



LEAST VALUE OF FUNCTION IN THIS DOMAIN

GREATEST VALUE OF FUNCTION IN THIS DOMAIN

LOCAL MINIMUM



BACK TO MAIN

Mathematics mind map

Analysis and approaches AHL

Analysis and approaches SL

Common content

Applications and interpretation SL

Applications and interpretation AHL

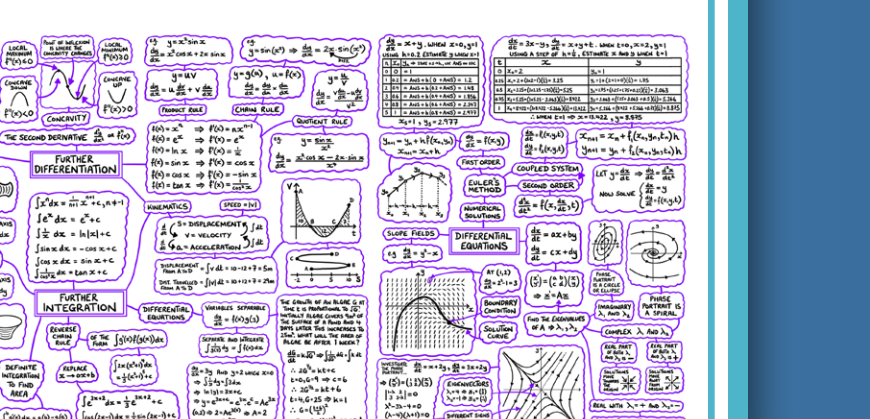
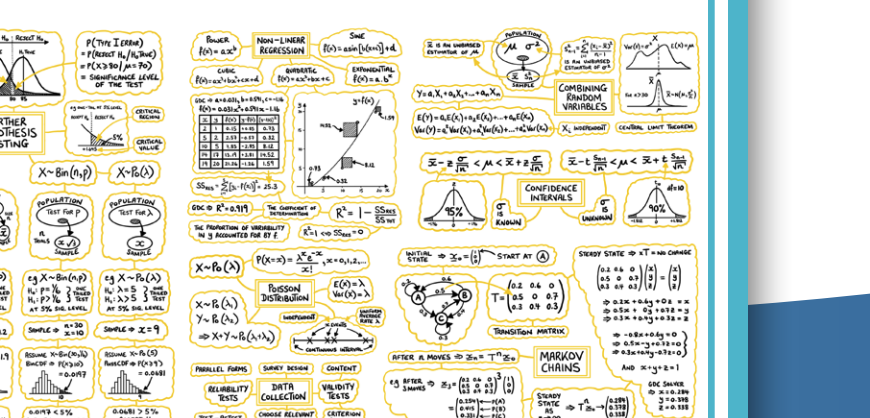
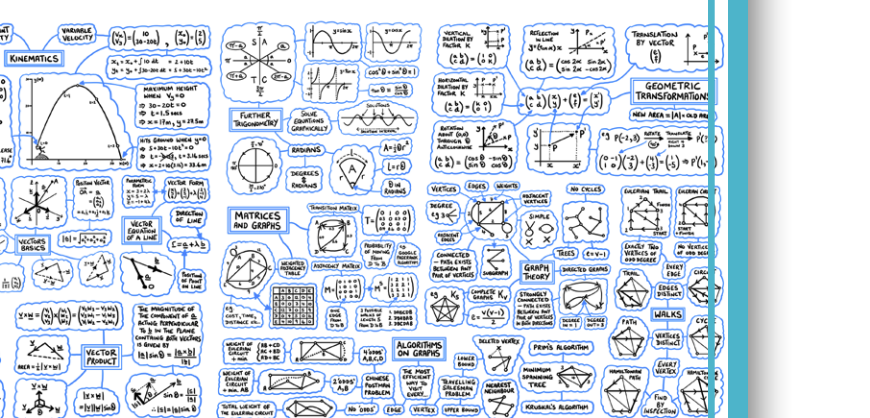
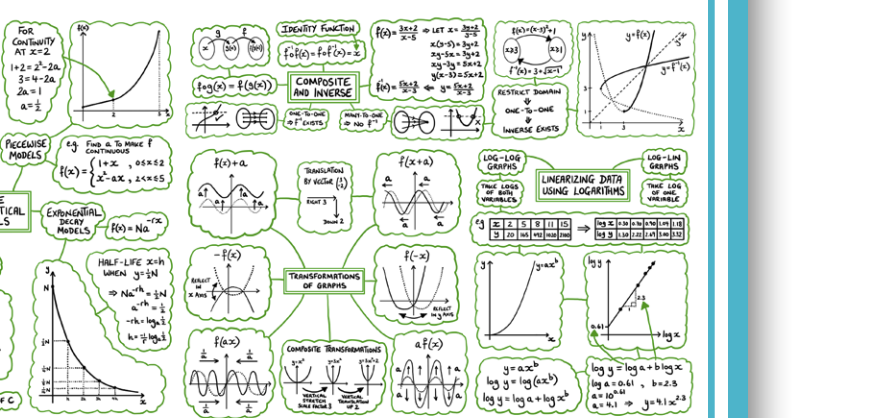
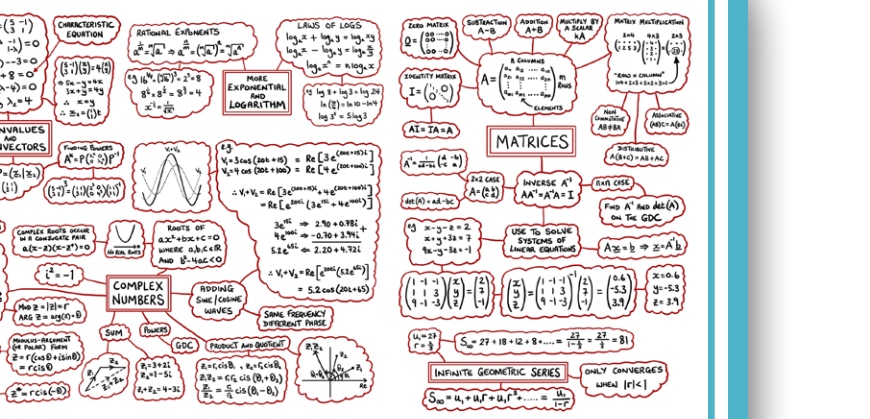
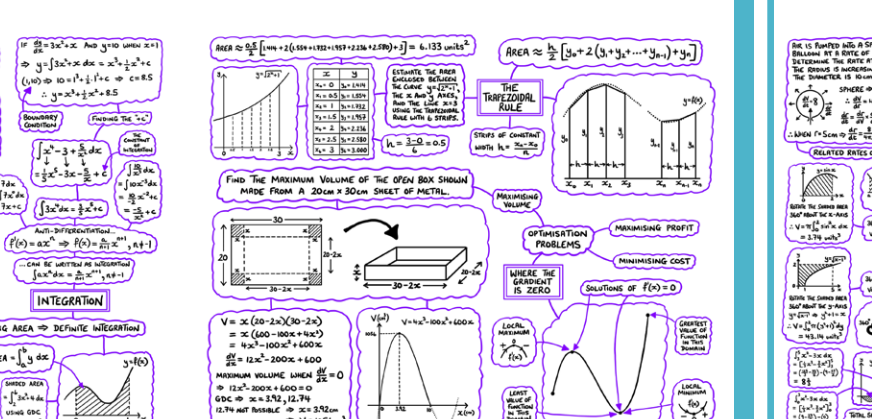
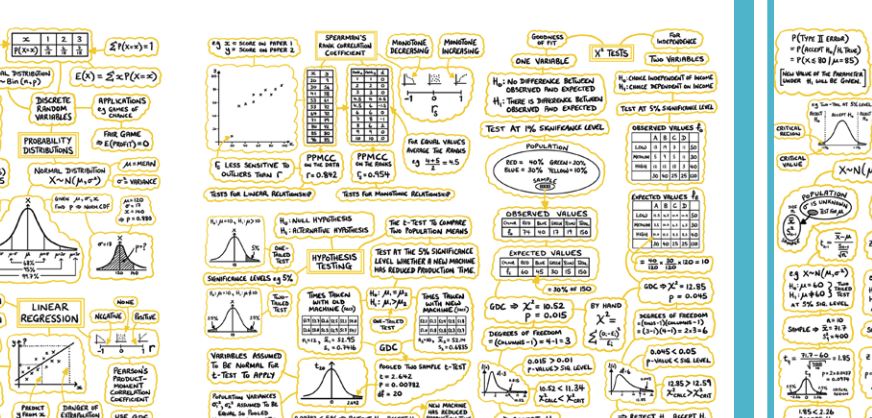
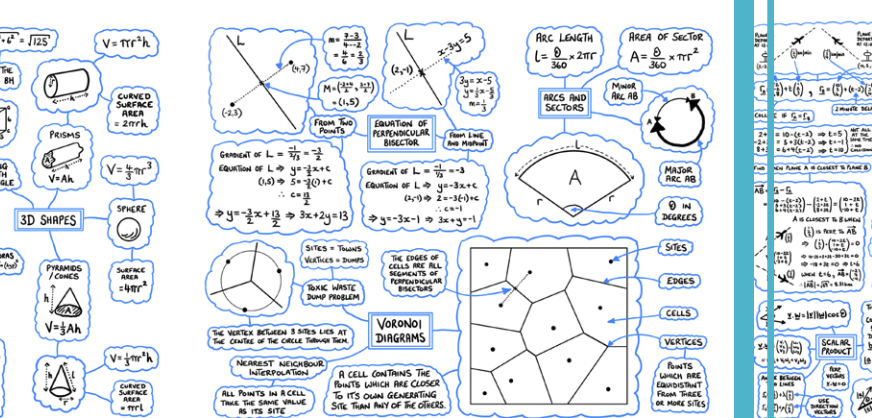
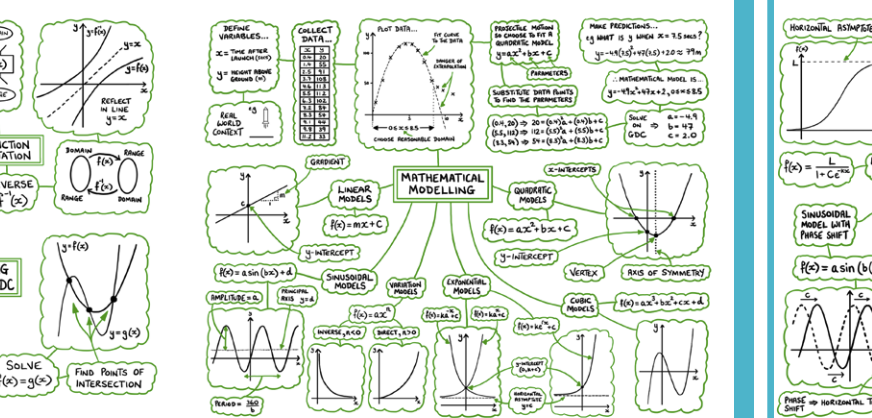
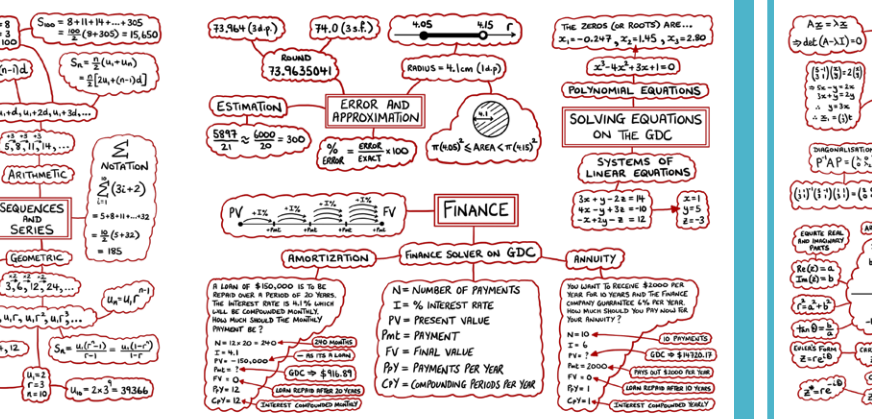
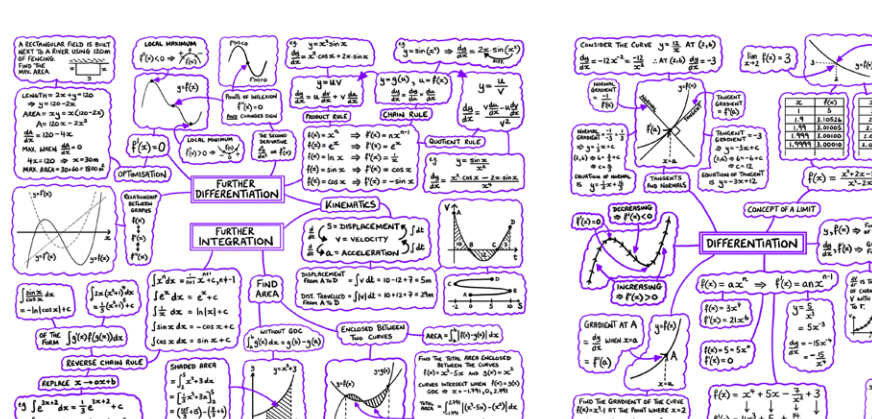
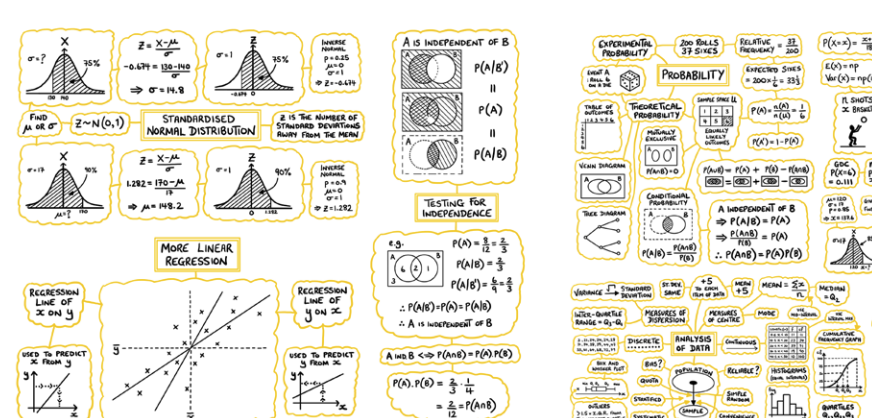
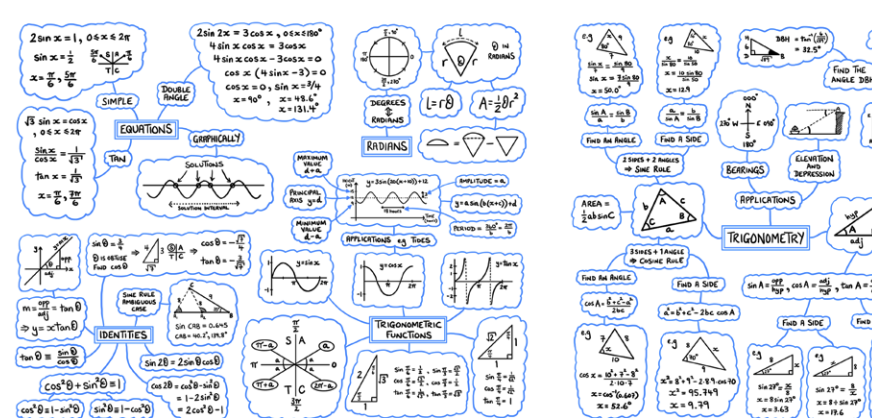
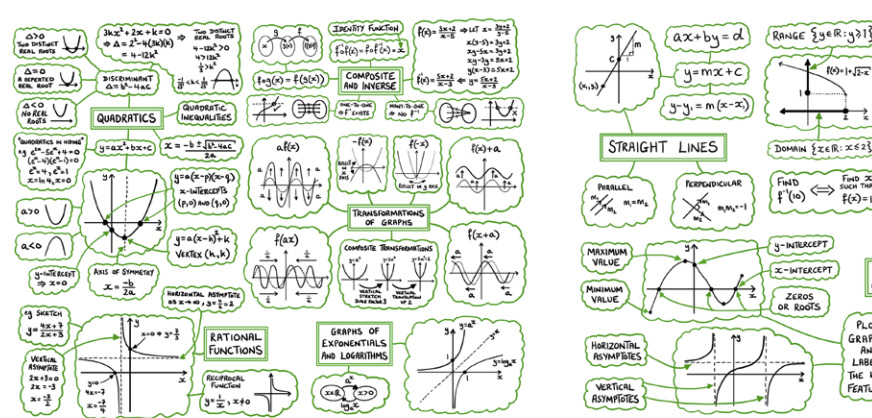
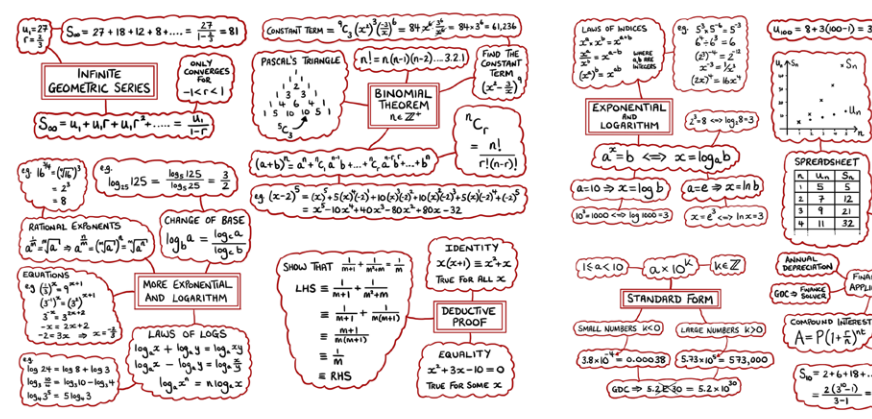
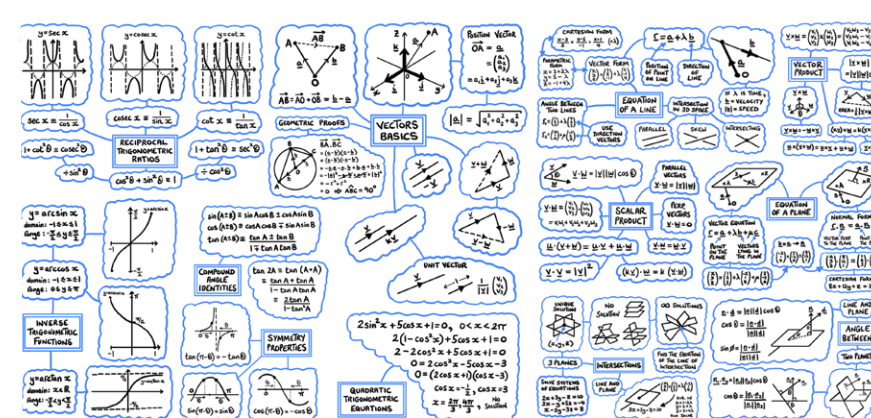
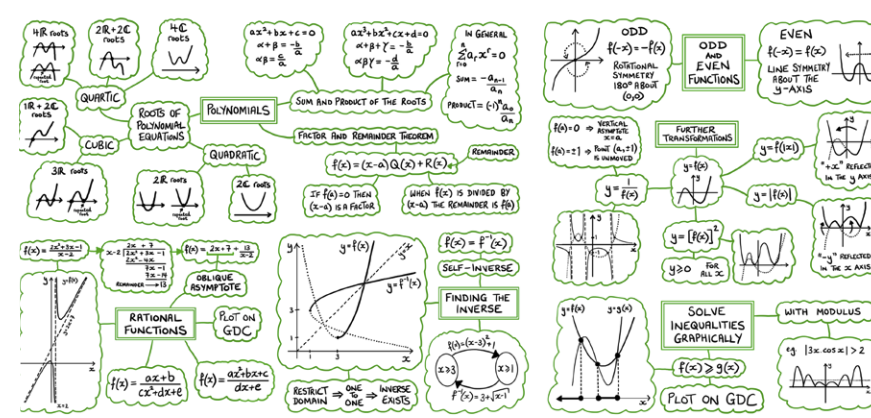
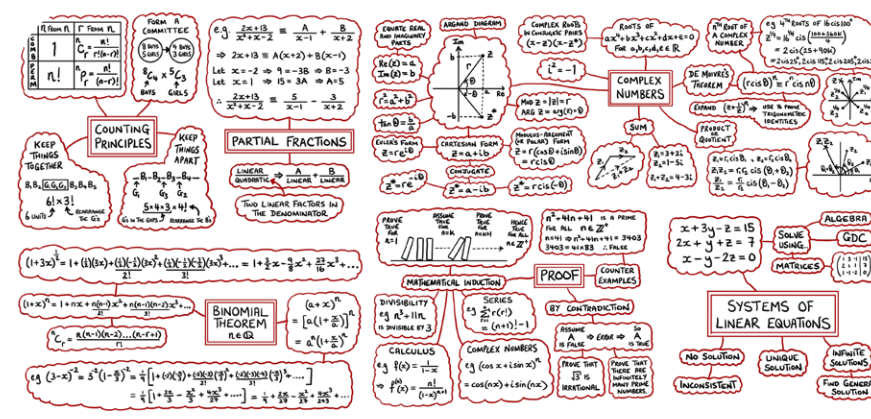
Number and algebra

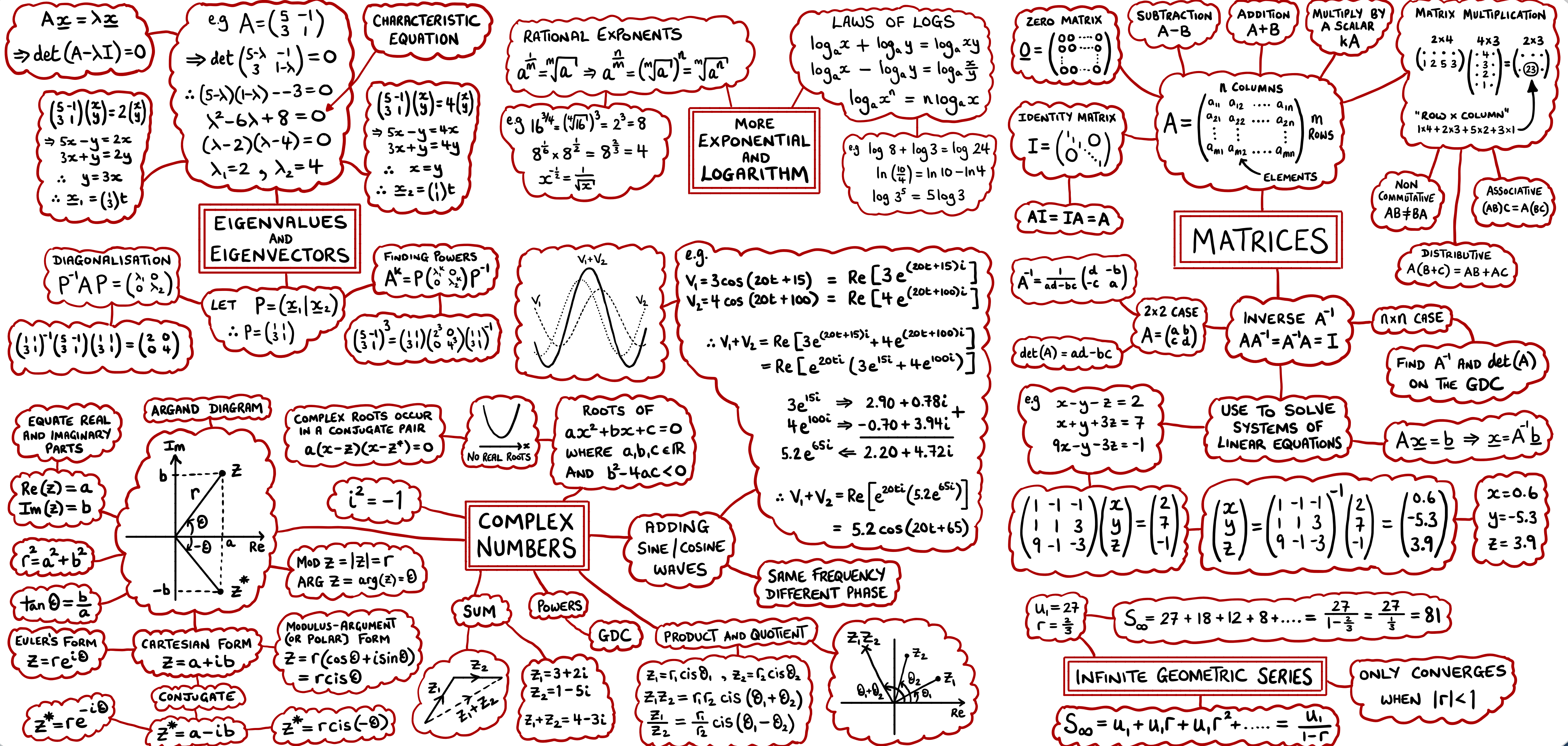
Functions

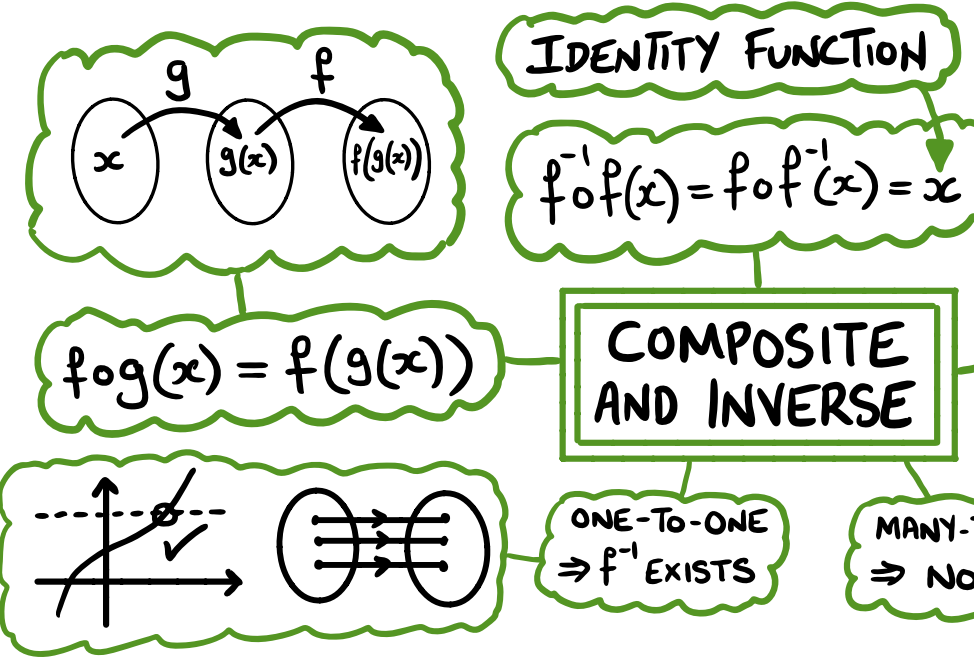
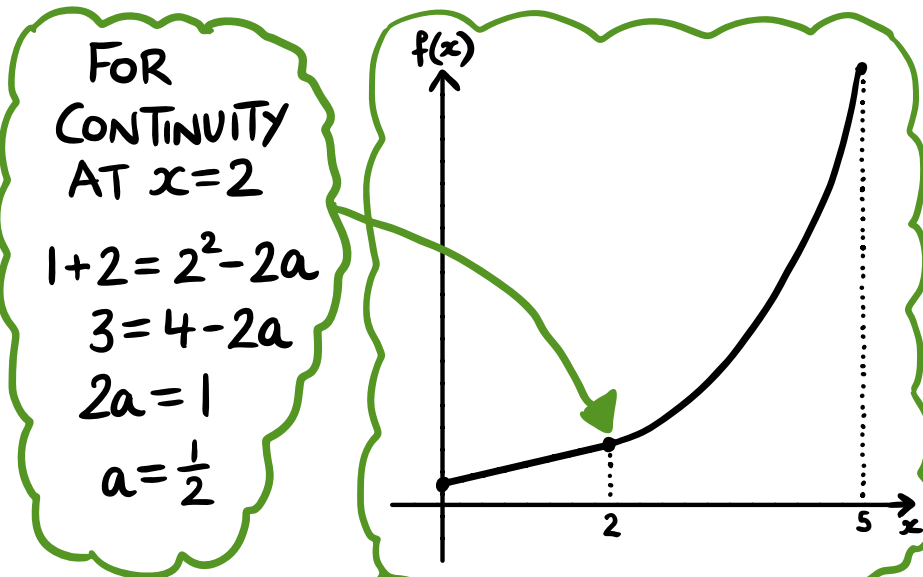
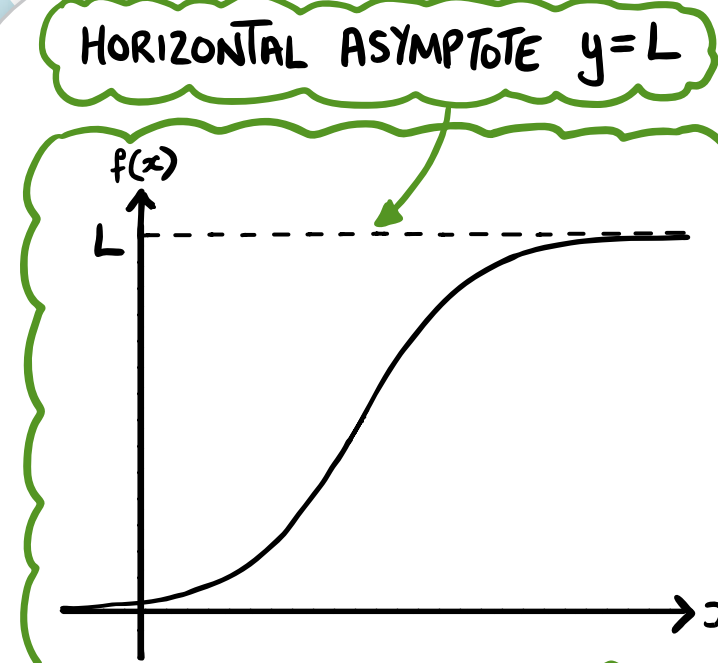
Geometry and trigonometry

Statistics and probability

Calculus







$f(x) = \frac{3x+2}{x-5} \Rightarrow$ LET $x = \frac{3y+2}{y-5}$

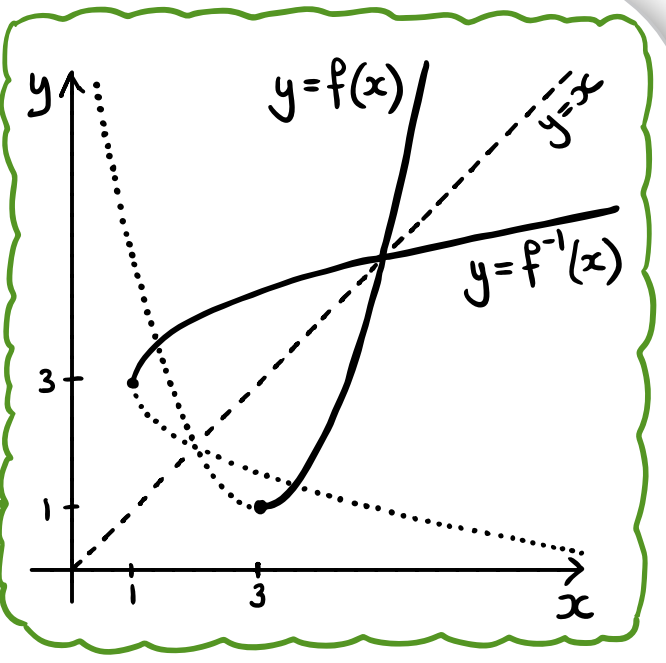
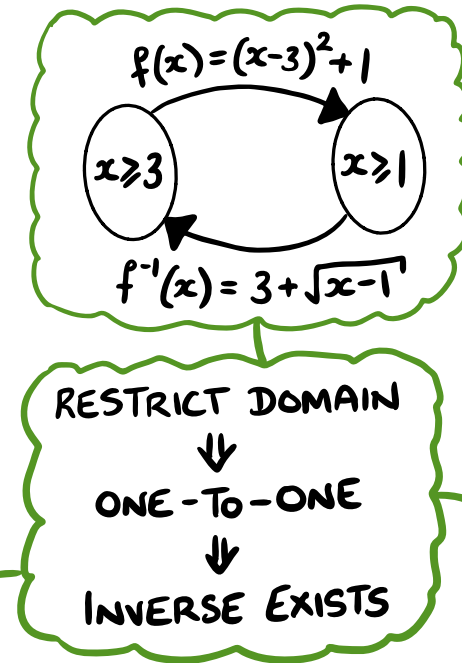
$$x(y-5) = 3y+2$$

$$xy-5x = 3y+2$$

$$xy-3y = 5x+2$$

$$y(x-3) = 5x+2$$

$$f^{-1}(x) = \frac{5x+2}{x-3} \Leftarrow y = \frac{5x+2}{x-3}$$



$f(x) = \frac{L}{1+Ce^{-kx}}$

LOGISTIC MODELS

PIECEWISE MODELS

e.g. FIND a TO MAKE f CONTINUOUS

$$f(x) = \begin{cases} 1+x, & 0 \leq x \leq 2 \\ x^2-ax, & 2 < x \leq 5 \end{cases}$$

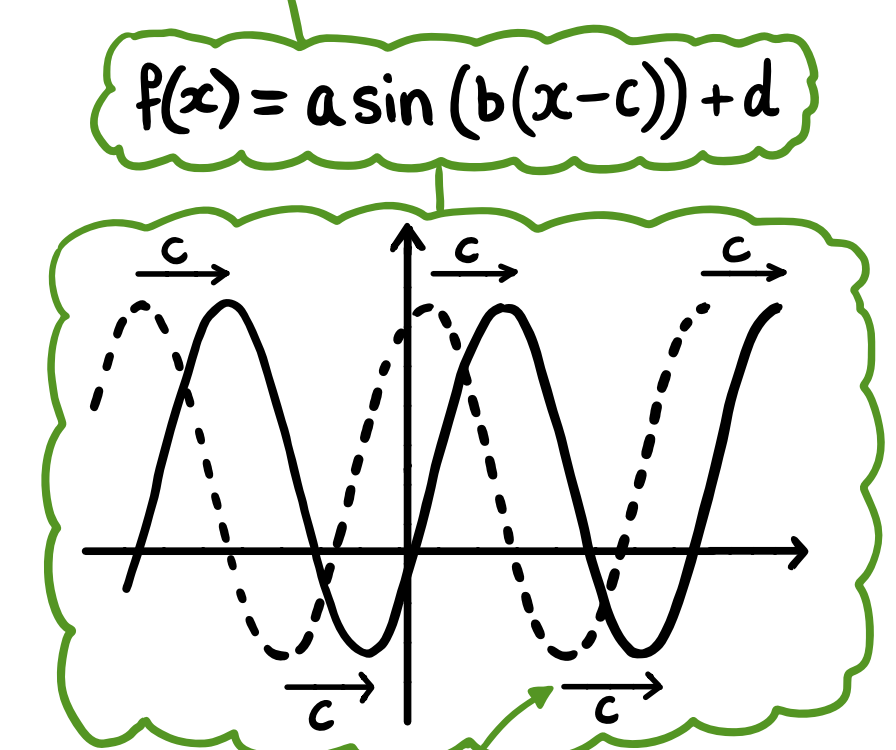
MORE MATHEMATICAL MODELS

SINUSOIDAL MODEL WITH PHASE SHIFT

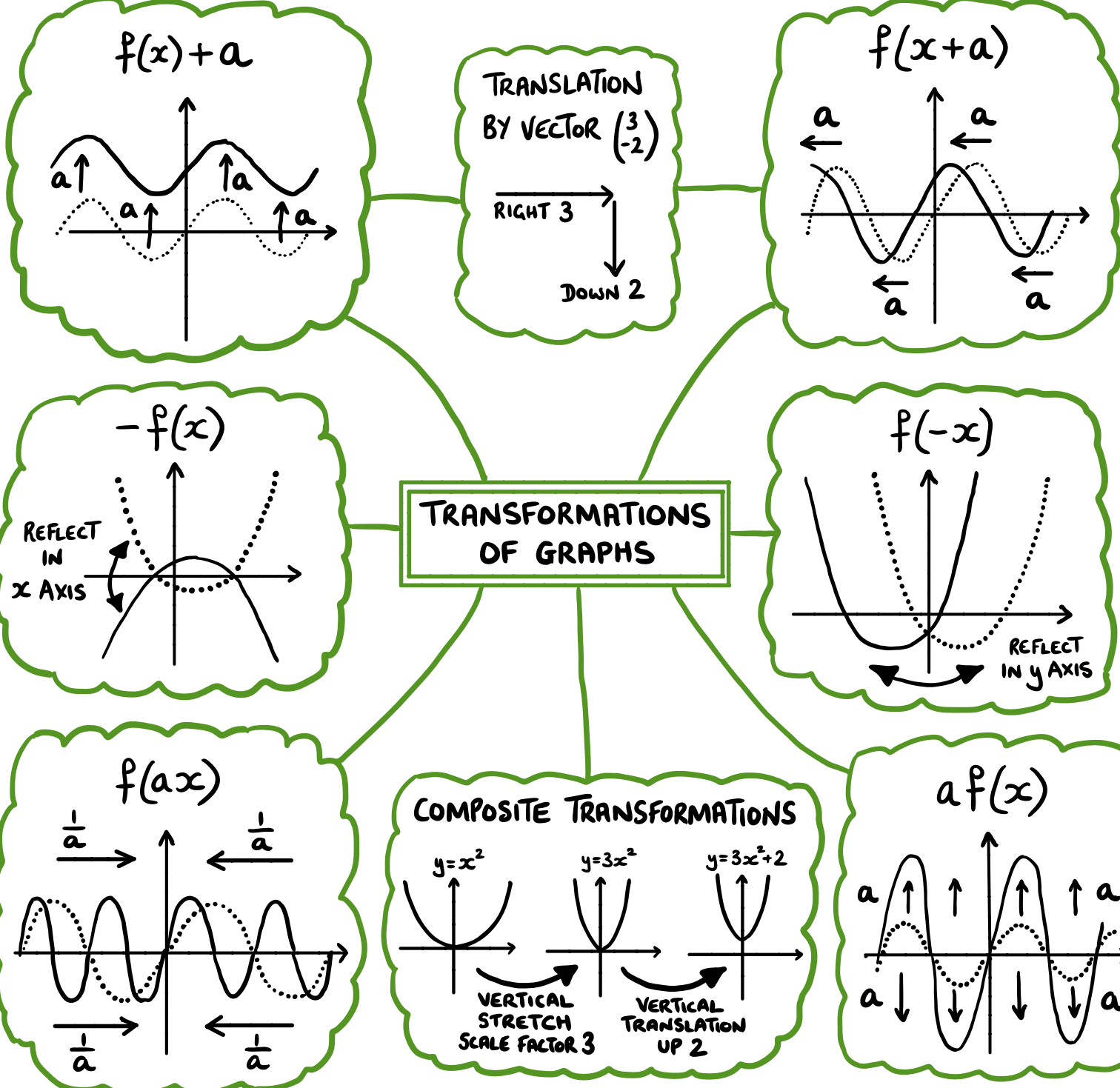
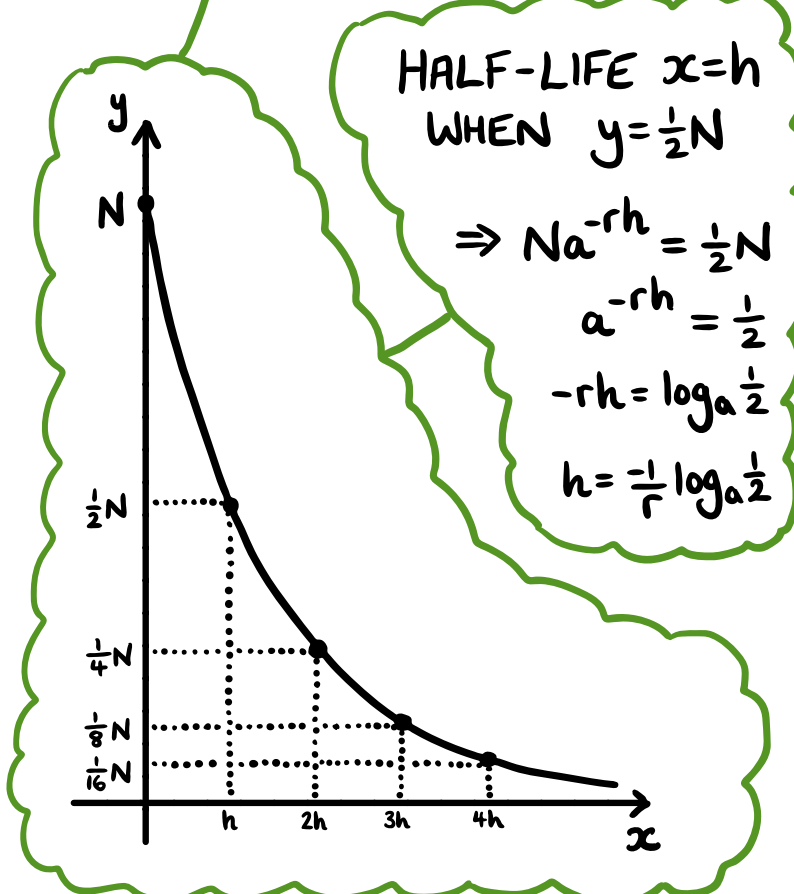
$$f(x) = a \sin(b(x-c)) + d$$

EXPONENTIAL DECAY MODELS

$$f(x) = Na^{-rx}$$



PHASE SHIFT $\Rightarrow$ HORIZONTAL TRANSLATION OF C



LOG-LOG GRAPHS

TAKE LOGS OF BOTH VARIABLES

LOG-LIN GRAPHS

TAKE LOG OF ONE VARIABLE

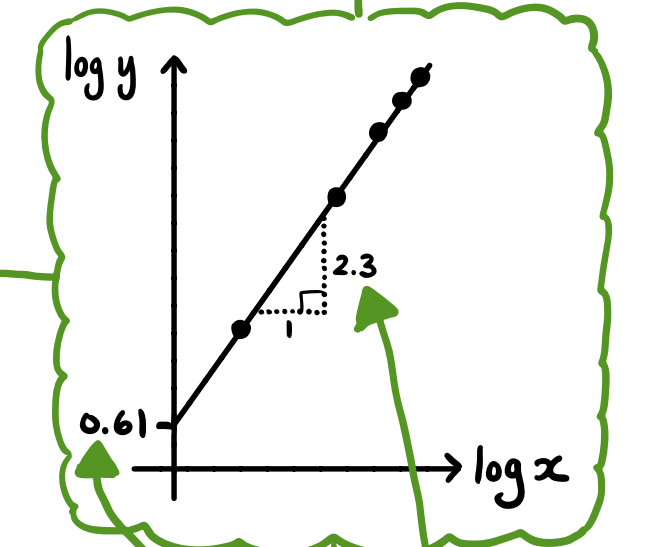
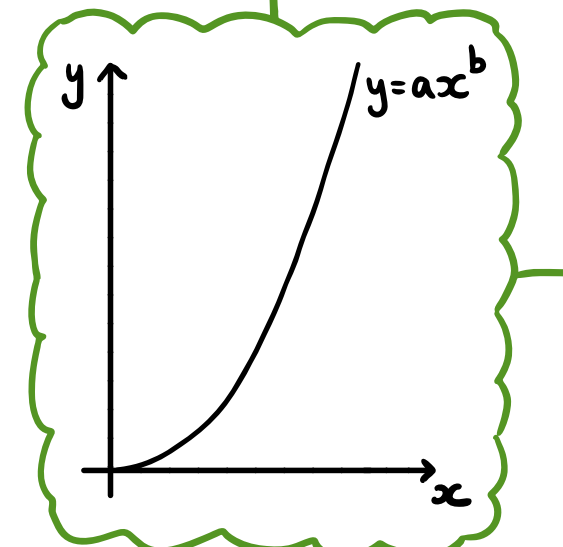
LINEARIZING DATA USING LOGARITHMS

e.g.

x	2	5	8	11	15
y	20	165	492	1020	2100

 $\Rightarrow$

$\log x$	0.30	0.70	0.90	1.04	1.18
$\log y$	1.30	2.22	2.69	3.00	3.32



$y = ax^b$

$$\log y = \log(ax^b)$$

$$\log y = \log a + b \log x$$

$\log y = \log a + b \log x$

$\log a = 0.61$, $b = 2.3$

$a = 10^{0.61}$

$a = 4.1 \Rightarrow y = 4.1x^{2.3}$

KINEMATICS

CONSTANT VELOCITY

PLANE A DEPARTS AT 12:00
PLANE B DEPARTS AT 12:02

PLANE A: $(2, -2, 8)$ $\frac{1}{2}$ km/min $(\frac{1}{2}, \frac{1}{2})$ km/min
PLANE B: $(10, 5, 6)$ $(\frac{1}{4}, \frac{1}{4})$ km/min

$\vec{r}_A = \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$, $\vec{r}_B = \begin{pmatrix} 10 \\ 5 \\ 6 \end{pmatrix} + (t-2) \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}$

COLLIDE IF $\vec{r}_A = \vec{r}_B$

$2+t = 10-(t-2) \Rightarrow t=5$
 $-2+2t = 5+(t-2) \Rightarrow t=-1$ NOT ALL AT THE SAME TIME
 $8+3t = 6+4(t-2) \Rightarrow t=10$ NO COLLISION

FIND WHEN PLANE A IS CLOSEST TO PLANE B

$\vec{AB} = \vec{r}_B - \vec{r}_A$
 $= \begin{pmatrix} 10-(t-2) \\ 5+3(t-2) \\ 6+4(t-2) \end{pmatrix} - \begin{pmatrix} 2+t \\ -2+2t \\ 8+3t \end{pmatrix} = \begin{pmatrix} 10-2t \\ 1+t \\ -10+t \end{pmatrix}$
 $\therefore |\vec{AB}| = \sqrt{(10-2t)^2 + (1+t)^2 + (-10+t)^2}$
 $= \sqrt{6t^2 - 58t + 201}$
 $\frac{d}{dt}(6t^2 - 58t + 201) = 0$
 $12t - 58 = 0 \Rightarrow t = 4\frac{5}{6}$ mins
 $\Rightarrow |\vec{AB}| = 7.8$ km

VARIABLE VELOCITY

$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 10 \\ 30-20t \end{pmatrix}$, $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$x_t = x_0 + \int 10 dt = 2 + 10t$
 $y_t = y_0 + \int 30-20t dt = 5 + 30t - 10t^2$

MAXIMUM HEIGHT WHEN $v_y = 0$
 $\Rightarrow 30 - 20t = 0 \Rightarrow t = 1.5$ secs
 $\Rightarrow x = 17$ m, $y = 27.5$ m

HITS GROUND WHEN $y = 0$
 $\Rightarrow 5 + 30t - 10t^2 = 0 \Rightarrow t = -0.158, t = 3.16$ secs
 $\Rightarrow x = 2 + 10(3.16) = 33.6$ m

ANGLE OF RELEASE $\theta = \tan^{-1}(\frac{30}{10}) = 71.6^\circ$

FURTHER TRIGONOMETRY

RADIANS

DEGREES $\leftrightarrow$ RADIANS

$A = \frac{1}{2} \theta r^2$
 $L = r\theta$
 θ IN RADIANS

SOLVE EQUATIONS GRAPHICALLY

$y = \sin x$, $y = \cos x$, $y = \tan x$

$\cos^2 \theta + \sin^2 \theta = 1$
 $\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$

TRANSITION MATRIX

$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0.3 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \\ 0.4 & 0.6 & 0 & 0 \end{pmatrix}$

PROBABILITY OF MOVING FROM D TO B

WEIGHTED ADJACENCY TABLE

	A	B	C	D	E
A	3	0	12	0	4
B	0	0	7	4	10
C	12	7	0	5	9
D	0	4	5	0	6
E	4	10	9	6	0

ONE EDGE FROM D TO B

3 POSSIBLE WALKS OF LENGTH 5 FROM D TO B

1. DABCD B
2. DBAB B
3. DBCD B

GEOMETRIC TRANSFORMATIONS

NEW AREA = $|A| \times$ OLD AREA

VERTICAL DILATION BY FACTOR K

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$

HORIZONTAL DILATION BY FACTOR K

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}$

REFLECTION IN LINE

$y = (\tan \alpha)x$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$

TRANSLATION BY VECTOR

$\begin{pmatrix} e \\ f \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

ROTATION ABOUT (0,0) THROUGH θ ANTICLOCKWISE

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

e.g. $P(-2, 3)$ ROTATE 90° TRANSLATE RIGHT 4 DOWN 3 $\Rightarrow P'(1, -5)$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

VECTORS BASICS

POSITION VECTOR

$\vec{OA} = \vec{a}$
 $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

PARAMETRIC FORM

$x = 3 + 2\lambda$
 $y = 5 - \lambda$
 $z = -1 + 4\lambda$

VECTOR FORM

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$

VECTOR EQUATION OF A LINE

$\vec{r} = \vec{a} + \lambda \vec{b}$

DIRECTION OF LINE

$\vec{b}$

POSITION OF POINT ON LINE

$\vec{r} = \vec{a} + \lambda \vec{b}$

UNIT VECTOR

$\frac{1}{|\vec{a}|} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

THE MAGNITUDE OF THE COMPONENT OF $\vec{a}$ IN THE DIRECTION OF $\vec{b}$ IS GIVEN BY

$|\vec{a}| \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

SCALAR PRODUCT

$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$

PERP. VECTORS $\vec{a} \cdot \vec{b} = 0$

ANGLE BETWEEN TWO LINES

$\vec{s}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{s}_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

$\cos \theta = \frac{|\vec{s}_1 \cdot \vec{s}_2|}{|\vec{s}_1| |\vec{s}_2|}$

MATRICES AND GRAPHS

ADJACENCY MATRIX

$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

WEIGHTED ADJACENCY TABLE

	A	B	C	D	E
A	3	0	12	0	4
B	0	0	7	4	10
C	12	7	0	5	9
D	0	4	5	0	6
E	4	10	9	6	0

ONE EDGE FROM D TO B

3 POSSIBLE WALKS OF LENGTH 5 FROM D TO B

1. DABCD B
2. DBAB B
3. DBCD B

GRAPH THEORY

VERTICES, **EDGES**, **WEIGHTS**

DEGREE e.g. 3

ADJACENT VERTICES

ADJACENT EDGES

CONNECTED - PATH EXISTS BETWEEN ANY PAIR OF VERTICES

SUBGRAPH

COMPLETE GRAPHS K_n

$e = \frac{v(v-1)}{2}$

NO CYCLES

TREES $e = v - 1$

DIRECTED GRAPHS

STRONGLY CONNECTED - PATH EXISTS BETWEEN ANY PAIR OF VERTICES IN BOTH DIRECTIONS

DEGREE IN = 1, **DEGREE OUT = 3**

WALKS

TRAIL, **CIRCUIT**, **PATH**, **CYCLE**

EVERY EDGE, **EDGES DISTINCT**, **VERTICES DISTINCT**, **EVERY VERTEX**, **FIND BY INSPECTION**

VECTOR PRODUCT

$\vec{v} \times \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$

AREA = $\frac{1}{2} |\vec{v} \times \vec{w}|$

THE MAGNITUDE OF THE COMPONENT OF $\vec{a}$ ACTING PERPENDICULAR TO $\vec{b}$ IN THE PLANE CONTAINING BOTH VECTORS IS GIVEN BY

$|\vec{a}| \sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$

$\sin \theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|}$

$\therefore |\vec{a}| = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}| \sin \theta}$

ALGORITHMS ON GRAPHS

WEIGHT OF EULERIAN CIRCUIT + MIN. AB

WEIGHT OF EULERIAN CIRCUIT + MIN. AC + BD

WEIGHT OF EULERIAN CIRCUIT + MIN. AD + BC

4 'ODDS' A, B, C, D

2 'ODDS' A, B

NO 'ODDS' EDGE

CHINESE POSTMAN PROBLEM

THE MOST EFFICIENT WAY TO VISIT EVERY...

TRAVELLING SALESMAN PROBLEM

LOWER BOUND, **UPPER BOUND**

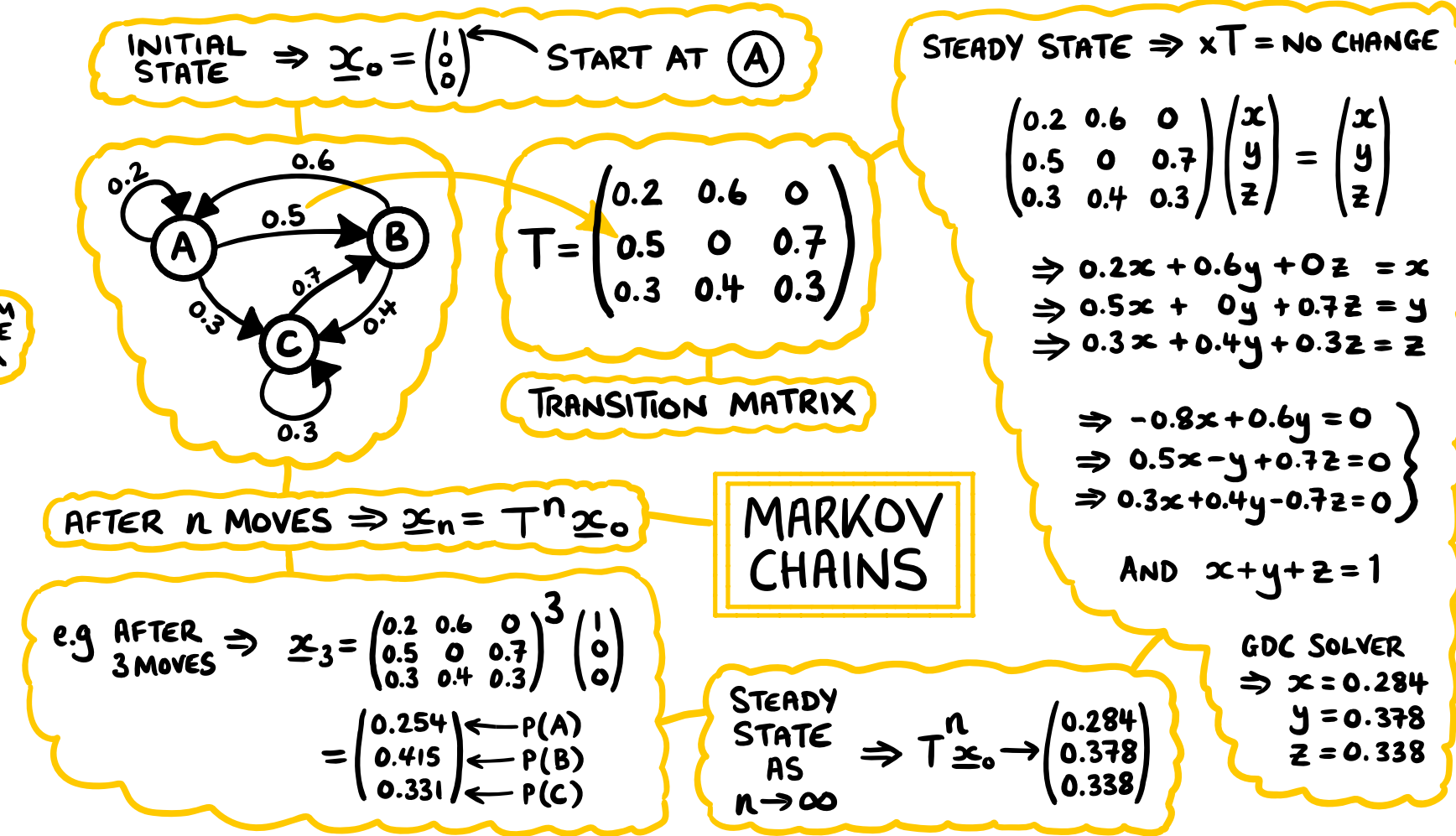
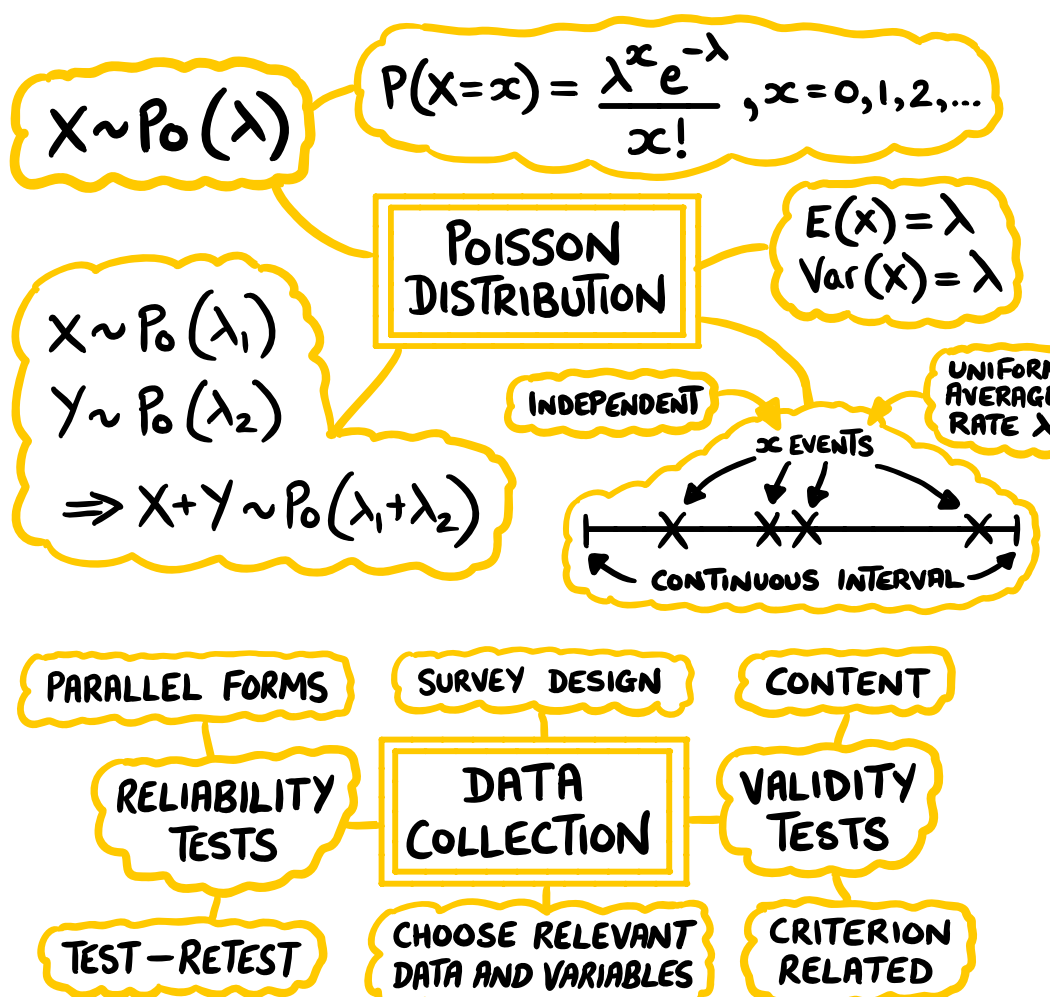
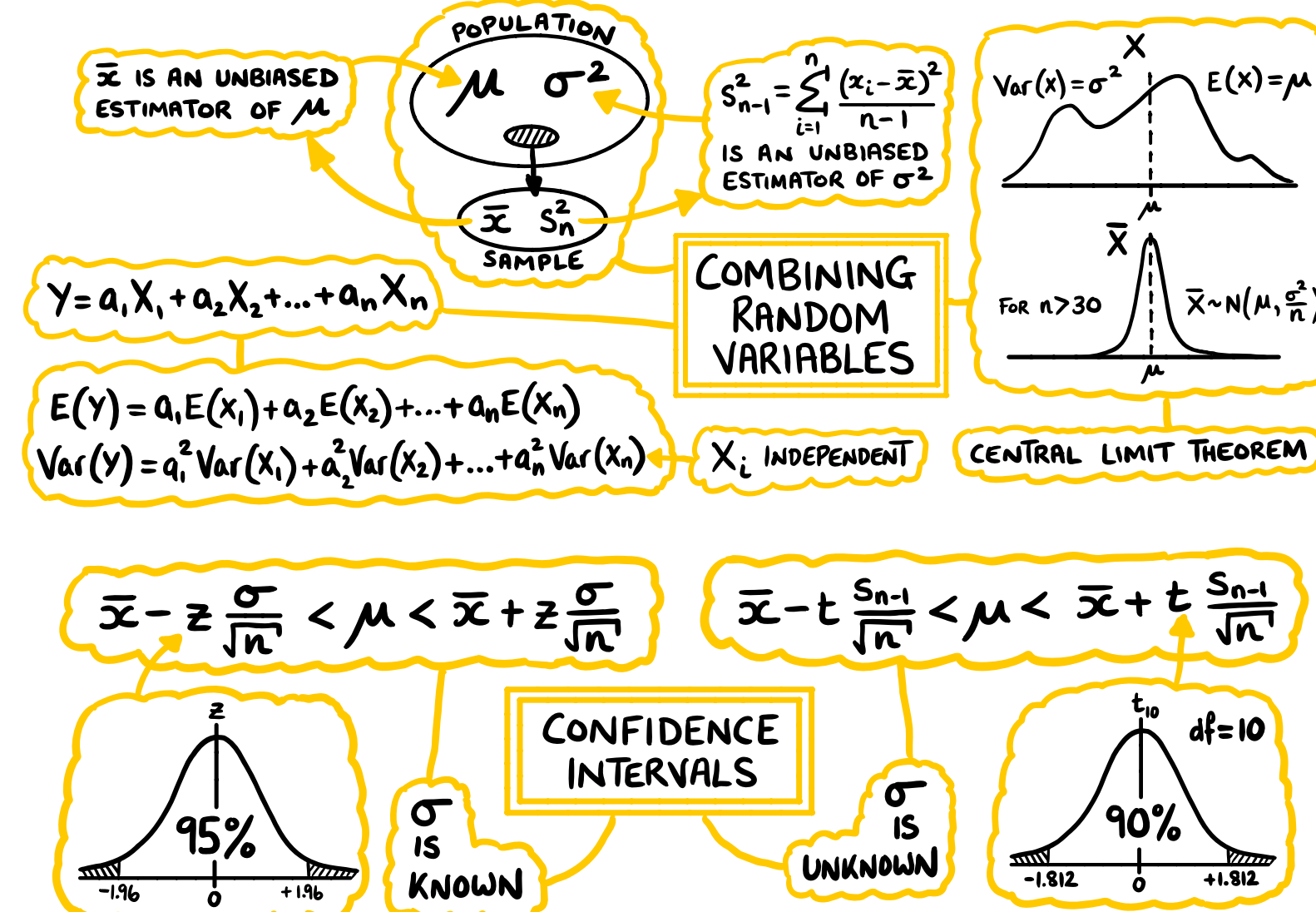
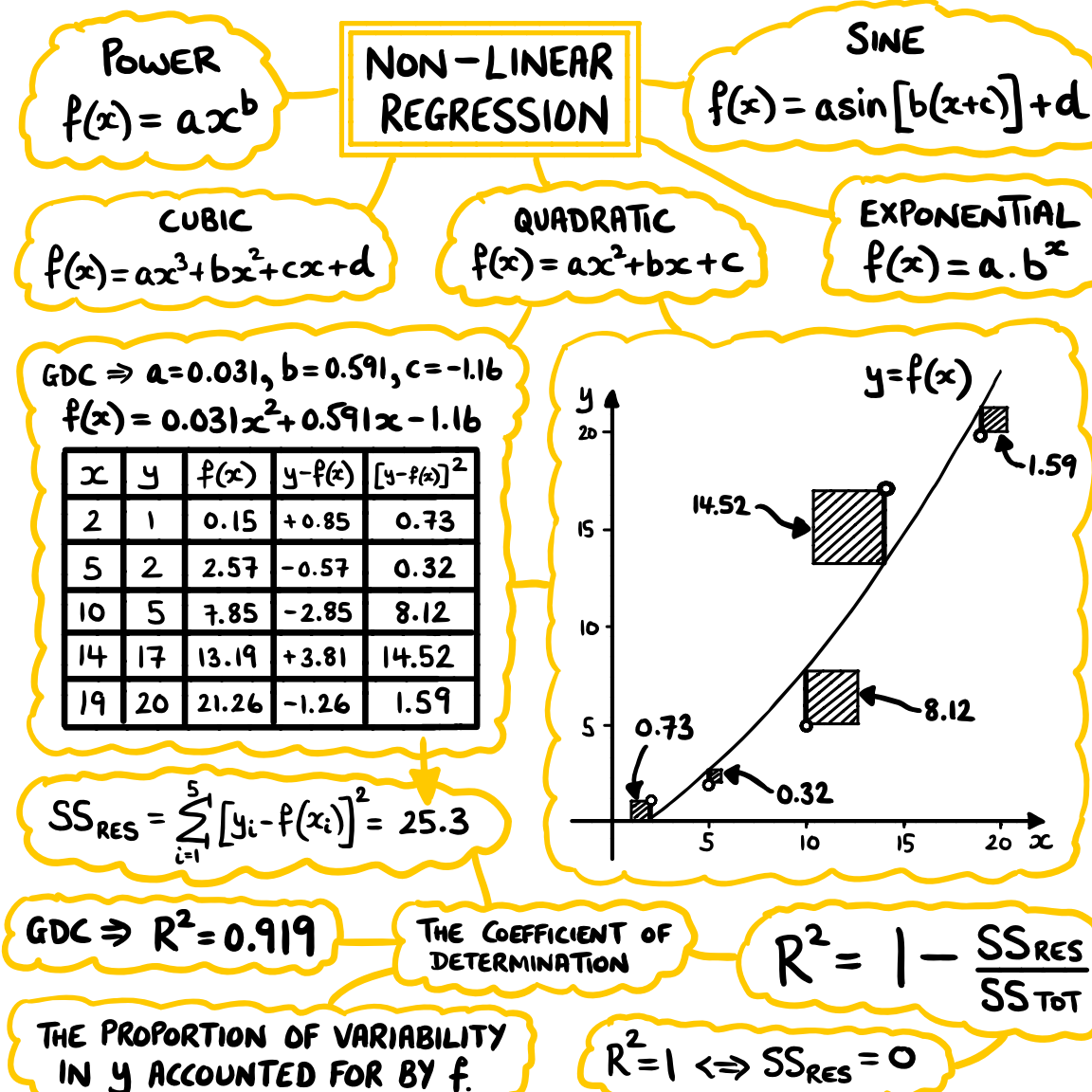
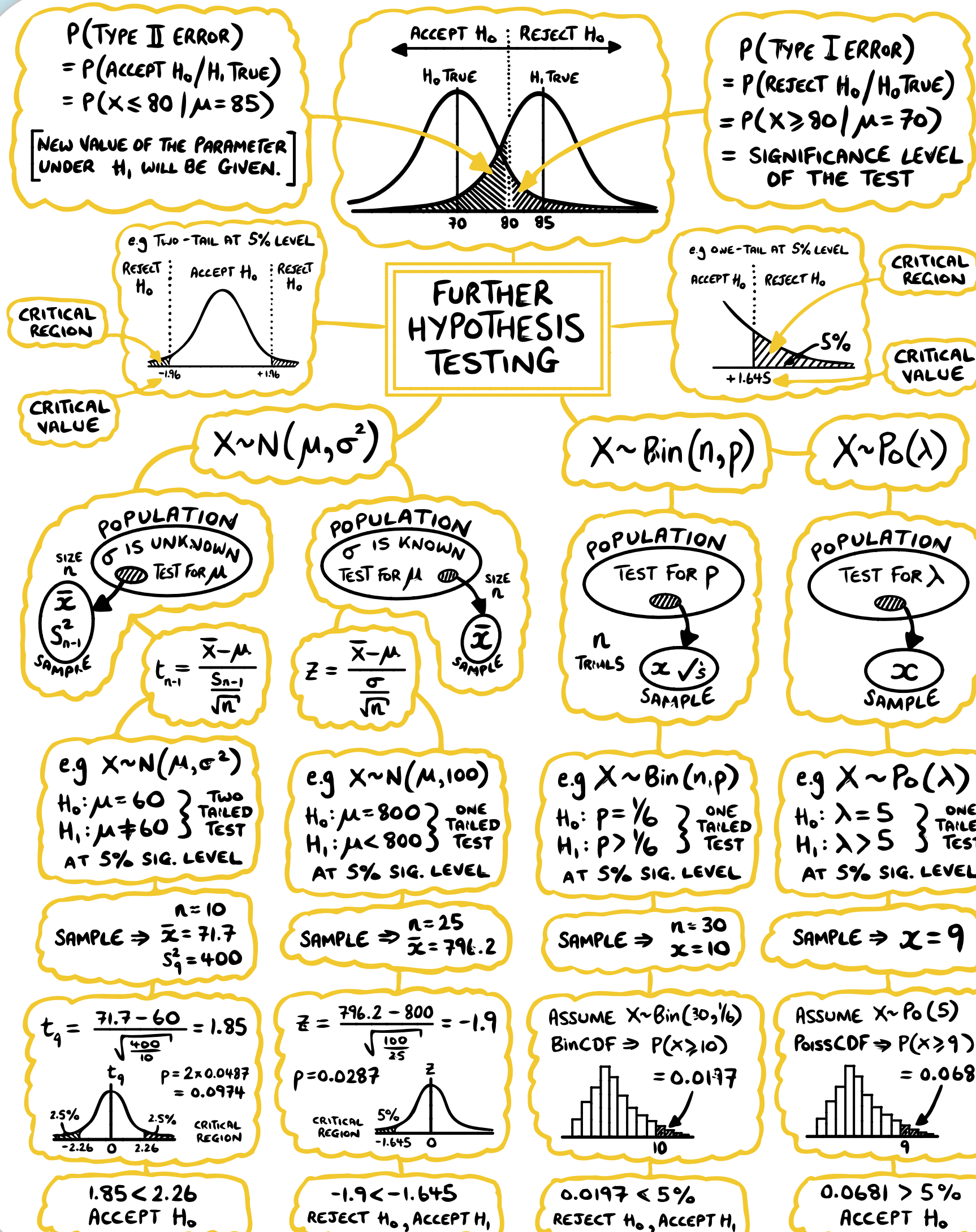
NEAREST NEIGHBOUR

PRIM'S ALGORITHM

MINIMUM SPANNING TREE

KRUSKAL'S ALGORITHM

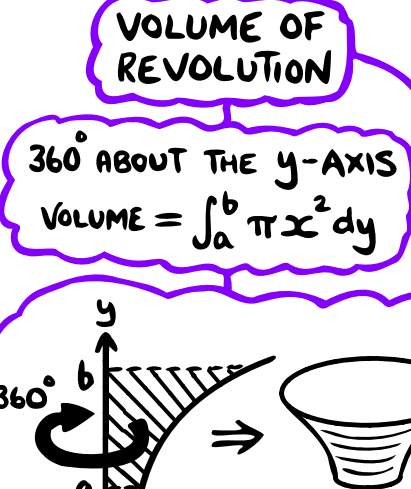
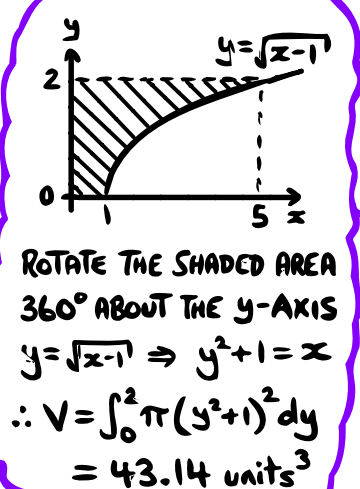
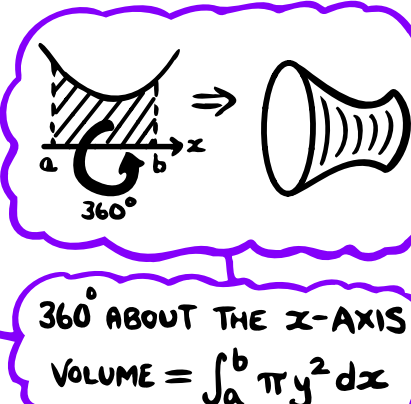
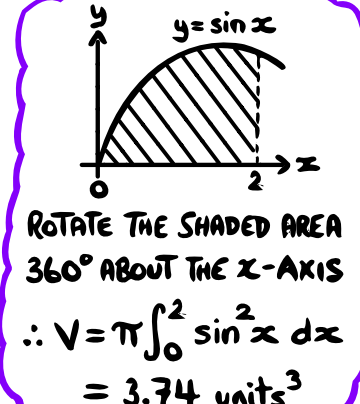
HAMILTONIAN PATH, **HAMILTONIAN CYCLE**



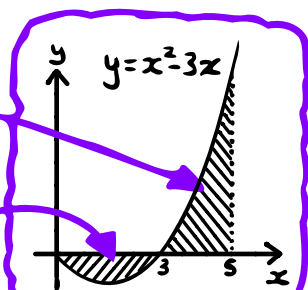
AIR IS PUMPED INTO A SPHERICAL BALLOON AT A RATE OF $8 \text{ cm}^3/\text{sec}$. DETERMINE THE RATE AT WHICH THE RADIUS IS INCREASING WHEN THE DIAMETER IS 10 cm .

SPHERE $\Rightarrow V = \frac{4}{3}\pi r^3$
 $\frac{dV}{dt} = 8$
 $\therefore \frac{dV}{dr} = 4\pi r^2$
 $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{8}{4\pi r^2}$
 $\therefore \text{WHEN } r = 5 \text{ cm} \Rightarrow \frac{dr}{dt} = \frac{8}{4\pi \cdot 5^2} = \frac{2}{25\pi} \text{ cm/s}$

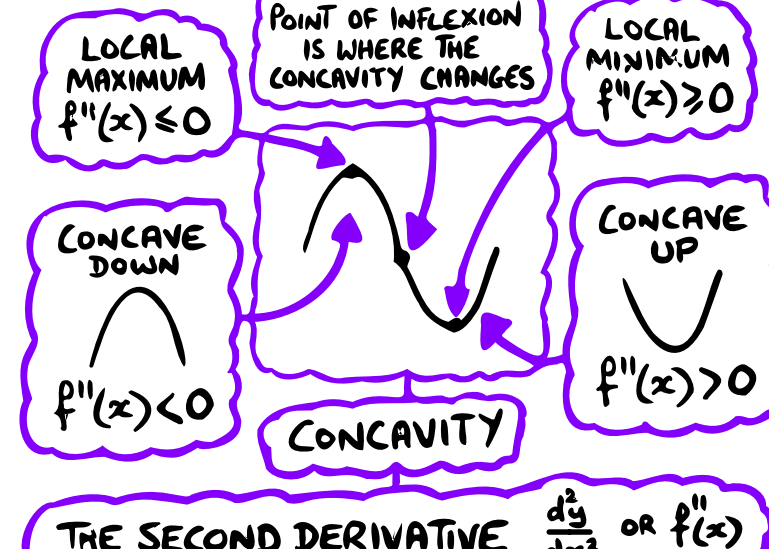
RELATED RATES OF CHANGE



$\int_3^5 x^2 - 3x \, dx$
 $= [\frac{1}{3}x^3 - \frac{3}{2}x^2]_3^5$
 $= (\frac{125}{3} - \frac{45}{2}) - (\frac{27}{3} - \frac{27}{2})$
 $= 8\frac{2}{3}$



$\int_0^3 x^2 - 3x \, dx$
 $= [\frac{1}{3}x^3 - \frac{3}{2}x^2]_0^3$
 $= (\frac{27}{3} - \frac{27}{2}) - (0 - 0)$
 $= -4\frac{1}{2}$



THE SECOND DERIVATIVE $\frac{d^2y}{dx^2}$ OR $f''(x)$

FURTHER DIFFERENTIATION

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, n \neq -1$
 $\int e^x \, dx = e^x + C$
 $\int \frac{1}{x} \, dx = \ln|x| + C$
 $\int \sin x \, dx = -\cos x + C$
 $\int \cos x \, dx = \sin x + C$
 $\int \frac{1}{\cos^2 x} \, dx = \tan x + C$

FURTHER INTEGRATION

REVERSE CHAIN RULE
 OF THE FORM $\int g(x)f(j(x)) \, dx$
 REPLACE $x \rightarrow ax+b$
 $\int 2x(x^2+1)^4 \, dx = \frac{1}{5}(x^2+1)^5 + C$
 $\int e^{3x+2} \, dx = \frac{1}{3}e^{3x+2} + C$
 $\int \cos(2x-1) \, dx = \frac{1}{2}\sin(2x-1) + C$

DEFINITE INTEGRATION TO FIND AREA

e.g. $y = x^2 \sin x$
 $\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

$y = uv$
 $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

PRODUCT RULE

e.g. $y = \sin(x^2) \Rightarrow \frac{dy}{dx} = 2x \cdot \cos(x^2)$

$y = g(u), u = f(x)$
 $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

CHAIN RULE

$y = \frac{u}{v}$
 $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

QUOTIENT RULE

e.g. $y = \frac{\sin x}{x^2}$
 $\frac{dy}{dx} = \frac{x^2 \cos x - 2x \sin x}{x^4}$

KINEMATICS

SPEED $= |v|$

$\frac{d}{dt} S = \text{DISPLACEMENT}$
 $\frac{d}{dt} v = \text{VELOCITY}$
 $\frac{d}{dt} a = \text{ACCELERATION}$

DISPLACEMENT FROM A TO D $= \int v \, dt = 10 - 12 + 7 = 5 \text{ m}$
 DIST. TRAVELLED FROM A TO D $= \int |v| \, dt = 10 + 12 + 7 = 29 \text{ m}$

DIFFERENTIAL EQUATIONS

VARIABLES SEPARABLE
 $\frac{dy}{dx} = f(x)g(y)$

SEPARATE AND INTEGRATE
 $\int \frac{1}{g(y)} \, dy = \int f(x) \, dx$

$\frac{dy}{dx} = 3y$ AND $y = 2$ WHEN $x = 0$
 $\Rightarrow \int \frac{1}{y} \, dy = \int 3 \, dx$
 $\Rightarrow \ln|y| = 3x + C$
 $\Rightarrow y = e^{3x+C} = e^{3x} \cdot e^C = Ae^{3x}$
 $(0, 2) \Rightarrow 2 = Ae^{3(0)} \Rightarrow A = 2$
 $\therefore y = 2e^{3x}$

THE GROWTH OF AN ALGAE G AT TIME t IS PROPORTIONAL TO $\sqrt{G}$. INITIALLY ALGAE COVERS 9 m^2 OF THE SURFACE OF A POND AND 4 DAYS LATER THIS INCREASES TO 25 m^2 . WHAT WILL THE AREA OF ALGAE BE AFTER 1 WEEK?

$\frac{dG}{dt} = k\sqrt{G} \Rightarrow \int \frac{1}{\sqrt{G}} \, dG = \int k \, dt$
 $\therefore 2\sqrt{G} = kt + C$
 $t = 0, G = 9 \Rightarrow C = 6$
 $\therefore 2\sqrt{G} = kt + 6$
 $t = 4, G = 25 \Rightarrow k = 1$
 $\therefore G = (\frac{t+6}{2})^2$
 $t = 7 \Rightarrow G = (\frac{7+6}{2})^2 = 42.25 \text{ m}^2$

$\frac{dy}{dx} = x + y$. WHEN $x = 0, y = 1$ USING $h = 0.2$ ESTIMATE y WHEN $x = 1$

n	x_n	y_n
0	0	1
1	0.2	$= \text{ANS} + h(0 + \text{ANS}) = 1.2$
2	0.4	$= \text{ANS} + h(0.2 + \text{ANS}) = 1.48$
3	0.6	$= \text{ANS} + h(0.4 + \text{ANS}) = 1.856$
4	0.8	$= \text{ANS} + h(0.6 + \text{ANS}) = 2.347$
5	1	$= \text{ANS} + h(0.8 + \text{ANS}) = 2.977$

$x_5 = 1, y_5 = 2.977$

$\frac{dx}{dt} = 3x - y, \frac{dy}{dt} = x + y + t$. WHEN $t = 0, x = 2, y = 1$ USING A STEP OF $h = \frac{1}{4}$, ESTIMATE x AND y WHEN $t = 1$

t	x	y
0	$x_0 = 2$	$y_0 = 1$
0.25	$x_1 = 2 + (3 \times 2 - 1)(\frac{1}{4}) = 3.25$	$y_1 = 1 + (2 + 1 + 0)(\frac{1}{4}) = 1.75$
0.5	$x_2 = 3.25 + (3 \times 3.25 - 1.75)(\frac{1}{4}) = 5.25$	$y_2 = 1.75 + (3.25 + 1.75 + 0.25)(\frac{1}{4}) = 3.063$
0.75	$x_3 = 5.25 + (3 \times 5.25 - 3.063)(\frac{1}{4}) = 8.422$	$y_3 = 3.063 + (5.25 + 3.063 + 0.5)(\frac{1}{4}) = 5.266$
1	$x_4 = 8.422 + (3 \times 8.422 - 5.266)(\frac{1}{4}) = 13.422$	$y_4 = 5.266 + (8.422 + 5.266 + 0.75)(\frac{1}{4}) = 8.875$

$\therefore \text{WHEN } t = 1 \Rightarrow x = 13.422, y = 8.875$

$y_{n+1} = y_n + h f(x_n, y_n)$
 $x_{n+1} = x_n + h$

FIRST ORDER

EULER'S METHOD

NUMERICAL SOLUTIONS

$\frac{dx}{dt} = f_1(x, y, t)$
 $\frac{dy}{dt} = f_2(x, y, t)$

COUPLED SYSTEM

SECOND ORDER

$\frac{d^2x}{dt^2} = f(x, \frac{dx}{dt}, t)$

$x_{n+1} = x_n + f_1(x_n, y_n, t_n)h$
 $y_{n+1} = y_n + f_2(x_n, y_n, t_n)h$

LET $y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2}$

NOW SOLVE $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = f(x, y, t) \end{cases}$

SLOPE FIELDS

e.g. $\frac{dy}{dx} = y^2 - x$

DIFFERENTIAL EQUATIONS

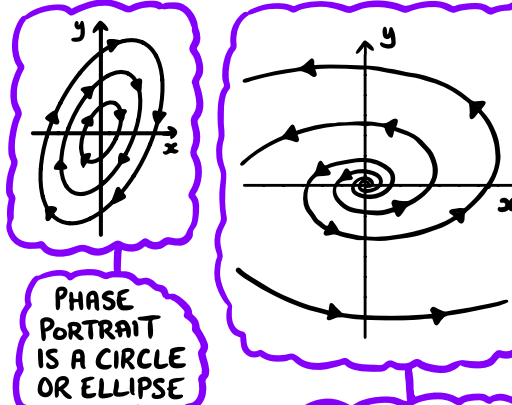
$\frac{dx}{dt} = ax + by$
 $\frac{dy}{dt} = cx + dy$

AT $(1, 2)$
 $\frac{dy}{dx} = 2^2 - 1 = 3$

BOUNDARY CONDITION

SOLUTION CURVE

FIND THE EIGENVALUES OF $A \Rightarrow \lambda_1, \lambda_2$



IMAGINARY λ_1 AND λ_2

PHASE PORTRAIT IS A SPIRAL

COMPLEX λ_1 AND λ_2

REAL PART OF BOTH λ_1 AND λ_2 IS $-$

SOLUTIONS MOVE TOWARDS THE ORIGIN

REAL PART OF BOTH λ_1 AND λ_2 IS $+$

SOLUTIONS MOVE AWAY FROM ORIGIN

REAL WITH $\lambda_1 = +$ AND $\lambda_2 = -$

PHASE PORTRAIT IS A SADDLE

INVESTIGATE THE PHASE PORTRAIT... $\frac{dx}{dt} = x + 2y, \frac{dy}{dt} = 3x + 2y$

$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
 $|1-\lambda \quad 2| = 0$
 $3-\lambda \quad 2 = 0$
 $(\lambda-4)(\lambda+1) = 0$
 $\lambda_1 = 4, \lambda_2 = -1$

EIGENVECTORS $\lambda_1 = 4 \Rightarrow \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
 $\lambda_2 = -1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

DIFFERENT SIGNS $\Rightarrow$ SOLUTIONS MOVE AWAY FROM ORIGIN

